Pricing in the Post Financial Crisis

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Contents

1 Introduction .................................................. 5

2 Pricing under the Full Collateralization ..................... 10
  2.1 Pricing under the full collateralization .................. 10
  2.2 Term structure modeling .................................. 13
    2.2.1 Dynamics of the forward collateral rate .......... 13
    2.2.2 Dynamics of the forward LIBOR-OIS spread .... 14
    2.2.3 Dynamics of the forward currency funding spread . 15
    2.2.4 Dynamics of the spot foreign exchange rate .... 15
    2.2.5 Summary of term structure dynamics .......... 16
  2.3 Curve construction in a single currency .............. 17
    2.3.1 Overnight Index Swap ................................ 17
    2.3.2 Interest rate swap .................................... 17
    2.3.3 Tenor swap .......................................... 18
    2.3.4 Calibration Example .................................. 18
  2.4 Collateralized FX ........................................... 19
    2.4.1 Collateralized FX forward .......................... 19
    2.4.2 FX option .............................................. 20
  2.5 Curve construction in multiple currencies ............ 21
    2.5.1 Cross currency swaps .................................. 21
    2.5.2 Historcal behavior of the funding spread .... 25
  2.6 OIS discounting and choice of collateral currency .... 25
  2.7 Embedded optionality in collateral agreements ........ 27
    2.7.1 Symmetric Cases .................................... 28
    2.7.2 Asymmetric Cases .................................... 29
  2.8 Standard Credit Support Annex and USD Silo .......... 31
    2.8.1 Discounting under the USD silo .................. 32
    2.8.2 FX forward under the USD silo .................. 32
  2.9 Fundamental instruments in the USD silo ................ 33
    2.9.1 IRS in the USD silo ................................ 33
    2.9.2 MtMCCS in the USD silo ........................... 33
    2.9.3 Curve construction in the USD silo .............. 34
    2.9.4 Heath-Jarrow-Morton framework in the USD silo . 35
    2.9.5 A possible adjustment for non-USD collateral .... 36
  2.10 Conclusion .................................................. 37
3 Pricing under Asymmetric and/or Imperfect Collateralization

3.1 Introduction

3.2 Generic Formulation

3.2.1 Fundamental Pricing Formula

3.3 Decomposing the Pre-default Value

3.3.1 Perfect and Symmetric Collateralization

3.3.2 Generic Situations

3.4 Perfect Collateralization

3.4.1 Symmetric Collateralization

3.4.2 Asymmetric Collateralization

3.5 Some Fundamental Instruments

3.5.1 Collateralized Zero Coupon Bond

3.5.2 Collateralized FX Forward

3.5.3 Overnight Index Swap

3.5.4 Cross Currency Swap

3.6 Numerical Studies for Asymmetric Collateralization

3.6.1 Asymmetric Collateralization for MtMCCOIS

3.6.2 Asymmetric Collateralization for OIS

3.7 General Implications of Asymmetric Collateralization

3.7.1 An implication for Netting

3.7.2 An implication for Resolution of Information

3.8 Imperfect Collateralization and CVA

3.9 Remarks on the collateral devaluation

3.10 Conclusions

4 Collateralized CDS

4.1 Introduction

4.2 Fundamental Pricing Formula

4.2.1 Setup

4.2.2 CDS Pricing

4.3 Financial Implications

4.4 Special Cases

4.4.1 3-party Case

4.4.2 4-party Case

4.5 Examples using a Copula

4.5.1 Framework

4.5.2 Numerical Examples

4.6 Conclusions

5 Concluding Remarks

A Appendix for Chapter 2

A.1 Proof of Proposition 1

A.2 Origin of the Funding Spread $y^{(i,k)}$ in the Pricing Formula

A.3 Calibration to swap markets

A.4 Details of Present Value Derivation in Sec. 3.6
Chapter 1

Introduction

The landscape of derivative modeling has experienced a radical change since the financial crisis in 2008. On the one hand, evaluation of the counterparty credit risk has become an unavoidable element for all the types of financial contracts. This is a rather natural consequence of significant number of credit events, well exemplified by a collapse of Lehman brothers, which was one of the most prestigious investment banks at that time. On the other hand, clean or benchmark pricing framework has also changed significantly. Here, explosion of various basis spreads, which were mostly negligible before the crisis, and the recognition of collateral cost have played the central role. In the thesis, we develop the pricing framework under the collateralization, in particular a framework of an interest rate model for the benchmark pricing under the full collateralization as well as the first order adjustment of remaining credit risk for generic situations. We also discuss a subtle but very important problem in credit derivative pricing using a collateralized credit default swap (CDS) as an example.

Collateralization in the over-the-counter (OTC) market has continued to grow at a rapid pace since the early 2000s. According to International Swaps and Derivatives Association (ISDA), about 70% of the trade volumes for all the OTC trades were collateralized at the end of 2011, which was merely 30% in 2003 [18]. As for the contracts of fixed income derivatives among the major financial institutions, nearly 90% of contracts are subject to collateral agreements. This trend is likely to accelerate due to the new regulations which give strong incentives to use the central counterparties (CCPs) by imposing severe capital charge for uncollateralized contracts as well as stringent collateral requirements for large portion of uncleared trades. The role of collateralization is mainly twofold: 1) reduction of the counterparty credit risk, and 2) change of funding costs of trades. The first point is rather obvious, but the importance of the second point has started being recognized only after the financial crisis.

Let us explain the gist of the second point using a schematic picture of a collateralized option contract in Figure 1.1. There are two firms A and B, and suppose that the firm A has just bought an option from the firm B. Then, on the A’s balance sheet, the option appears as a new item of financial asset. On the other hand, the firm A receives the collateral from the firm B, which is recognized as an item of financial liability. Under the full collateralization, where the amount of collateral is adjusted to have the equal value with the option throughout the contract, one can interpret that the new item in the A’s asset, or the option, is financed by the corresponding liability, i.e., the collateral. The firm
A, which is the receiver of the collateral, has to pay the fee on the posted collateral. It is called the “collateral rate”. Then, it is clear that the funding cost of the option is given by the corresponding collateral rate. Although one needs to handle more complicated accounting rules and portfolio of all the contracts under the netting agreement instead of a single option, the above concept still applies.

If a contract is not collateralized, its funding cost is directly linked to the firm’s overall funding rate. If the firm can raise external fund by LIBOR, then it can be interpreted as the firm’s funding rate. On the other hand, if the contract is collateralized by cash, which actually accounts more than 80% of collateral in OTC market [18], the collateral rate is usually given by the overnight rate of the collateral currency. The difference of the two rates, denoted by LIBOR-OIS spread, skyrocketed at the Lehman default and has been maintaining elevated levels since then, which was just a few basis point before the crisis. It should be now intuitively clear that LIBOR discounting does not make sense for collateralized contracts. Since LIBORs are still being used as the most popular reference rates in the financial contracts, one has to deal with multiple curves even in the single currency contracts.

As a natural consequence, multi-curve modeling has suddenly become popular in academia after the financial crisis. However, in the financial industry, the multi-curve modeling framework has existed for quite many years due to a quite different reason. In the late 1990s when Japanese bubble burst, Japanese financial firms were required to pay extra premium to US firms to fund USD cash. The cost arose from a negative cross currency swap (CCS) basis spread. A schematic picture of CCS is given in Figure 1.2 where the external funding rates of the two firms are assumed to be the LIBOR of the each home currency. There, a Japanese and a US financial firms are exchanging JPY and USD loans. Although the US firm receives USD-LIBOR from the Japanese firm, it pays JPY-LIBOR+$X_{ccs}$ to the Japanese firm on the borrowed JPY loan. At that time, the basis spread denoted by $X_{ccs}$ was around ($-40 \sim -50$)bps. Then, it is easy to see that the JPY funding cost for the US firm was lower than JPY-LIBOR. On the other hand, the USD funding cost for the Japanese firm was higher than USD-LIBOR since it had to compensate the lower return from the JPY lending than its borrowing cost from the local market. Suppose that a subsidiary of the US firm operating in Japan can lend JPY loan with JPY-LIBOR flat to a local cooperate. Since its JPY funding cost is lower than JPY-LIBOR, this financial firm needs to recognize a positive return from the contract. A simple JPY-LIBOR discounting method cannot properly reflect the reality. To handle the
Figure 1.2: A schematic picture of USDJPY cross currency swap between the two firms with LIBOR external funding.

problem, financial firms developed a two-curve model for JPY contracts in the late 1990s, where the one curve is used specifically to discount the JPY cash flows and the other is to derive forward JPY-LIBOR.

The cross currency basis spreads are still very much relevant for the financial market after the crisis. In Figures 1.3 and 1.4, the historical behavior of USDJPY and EURUSD CCS basis spreads are given for the period of 2005 May to 2010 July. One can see that the spreads have widened significantly and become quite volatile after the crisis. Since a CCS spread represents the difference of the funding cost between the two currencies, its significant size indicates the choice of collateral currency does affect the funding cost of the associated contract. Another important aspect of CCS can be understood through its relation to the FX forward. For example, if we combine a USDJPY CCS by USD as well as JPY standard interest rate swaps (IRS) with the same maturity, we obtain USD fixed vs JPY fixed swap, which is basically equivalent to the USDJPY FX forward. Thus, if a model does not allow calibration to the relevant set of CCSs, there is no hope that it recovers the market consistent FX forwards.

Figure 1.3: Historical behavior of USDJPY CCS basis spreads. Source:Bloomberg.
The above arguments make it clear that one needs to extend the traditional pricing framework substantially even under the full collateralization although it makes the counterparty credit risk negligible in many relevant situations. In order to construct the model which fits well to the market, various basis spreads are required to be dealt with explicitly. The role of risk-free rate needs to be rethought, too. The financial crisis made the naive assumption "LIBOR \approx \text{risk-free rate}" totally useless. Next obvious candidate is overnight rate, but we will see that this is not a good enough approximation since it completely fails to explain the market CCS basis spreads. Government bonds do not work well, either. They receive different liquidity premium for each maturity due to, for examples, different issuing volumes, fragmented types of investors, etc. In addition, they are often under particular regulations on risk management and taxation to incentivize the investors to hold them for smooth fulfillment of the financing needs of the governments. Thus, there should remain significant ambiguities in price of assets if the model requires explicit term structure of unobservable (or, at least very difficult to estimate) risk-free rate. We emphasize that the new framework is free from the above problem. Under the full collateralization, the model can be built solely from the market observable interest rates, such as OIS, LIBOR and CCS basis spreads. Furthermore, this feature basically holds true (except rather indirect and mild dependence) even in the situation with non-negligible credit risk due to the imperfect collateralization.

**Organization of the Thesis**

The organization of the thesis is as follows: In Chapter 2, we discuss the interest rate modeling framework under the full collateralization in a multi-currency environment. Firstly, we discuss the pricing formula under the collateralization including foreign collateral currency, and explain the no-arbitrage dynamic term structure modeling in Heath-Jarrow-Morton (HJM) framework. We then explain the curve construction procedures, which specifies the initial condition for the HJM model. We also discuss the importance of the choice of collateral currency and the possible embedded cheapest-to-deliver (CTD) optionality in some collateral agreements. Finally, we briefly discuss on the standard
credit support annex (SCSA) with a particular attention to the emerging currencies to be allocated to the USD silo. This chapter is mainly based on Refs. [10, 11, 12, 14].

In Chapter 3, we extend the framework to more generic situations with asymmetric and/or imperfect collateralization taking into account the credit risk explicitly both for the counterparty and the investor. Firstly, we rediscover the results of Chapter 2 as a special case of full collateralization in this more generic setup. We provide the first-order decomposition formula around the benchmark clean price. This formula should be more practical than the traditional definition of credit value adjustment (CVA) around the "risk-free" price. This is because, as we have argued, one cannot observe uncontroversial risk-free price in the market any more but only the benchmark price under the full (i.e. perfect) collateralization. We shall also see that there exists another adjustment factor called "collateral cost adjustment" (CCA), which explains the change of collateral cost from that of full collateralization. This new term is shown to play a significant role when one deals with the issues of unilateral collateralization that is common for trades between private financial firms and sovereigns or supranational entities, for examples. We also use these results to study the implication of sophistication in collateral management for financial firms. The contents of this chapter are from Ref. [13].

Lastly, in Chapter 4, we study the pricing of continuously collateralized CDS with variation margin. We make use of the "survival measure" to derive the pricing formula in a straightforward way. As a result, we shall see that there exists irremovable trace of the counterparty as well as the investor in the price of CDS through their default dependence even under the perfect collateralization, although the hazard rates of the two parties are totally absent from the pricing formula. This finding poses a very difficult problem for the central counterparties to carry out "proper" mark-to-market and risk management. This chapter is based on Ref. [15].

Chapter 5 provides the brief summary and remarks of the thesis. We discuss remaining and related issues which may become crucial in the financial market in the coming years. We also make a brief comment on the current progress as well as the future direction of the own research.
Chapter 2

Pricing under the Full Collateralization

2.1 Pricing under the full collateralization

Firstly, we review Fujii, Shimada & Takahashi (2010) [10], a generic pricing formula under the full collateralization. Let us make the following simplifying assumptions about the collateral contract.

1. Full collateralization (zero threshold) by cash.
2. The collateral is adjusted continuously with zero minimum transfer amount.

This means that the party who has negative mark-to-market posts the equal amount of cash collateral to the counterparty, and this is done continuously until the contract expires. Actually, daily margin call is becoming popular, and that is a market standard at least among major broker-dealers [18]. This observation allows us to see the above assumptions a reasonable proxy for the reality. One might consider there is no counterparty risk remains under the above assumption. In fact, however, this is not always the case, if there exists a sudden jump of the underlying asset and/or the collateral values at the time of counterparty default. This is the so-called “gap risk”. If this is the case, the remaining risk should be taken into account as a part of credit risk valuation adjustment (CVA). In this chapter, we assume no counterparty risk remains and focus on the clean price. For the interested readers, we refer to [13] to handle more generic situations with imperfect collateralization and the price decomposition around the clean part.

We consider a derivative whose payoff at time $T$ is given by $h^{(i)}(T)$ in terms of currency $(i)$. We suppose that currency $(j)$ is used as the collateral for the contract. Let us introduce an important spread process:

$$
y^{(j)}(t) = r^{(j)}(t) - c^{(j)}(t), \tag{2.1.1}
$$

where $r^{(j)}$ and $c^{(j)}$ denote the risk-free interest rate and the collateral rate of the currency $(j)$, respectively. A common practice in the market is to set $c^{(j)}$ as the overnight (ON) rate of currency $(j)$. Economically, the spread $y^{(j)}$ can be interpreted as the dividend yield from the collateral account of currency $(j)$ from the view point of a collateral receiver.
On the other hand, from the view point of a collateral payer, it can be considered as a collateral funding cost. Of course, the return from risky investments, or the borrowing cost from the external market can be quite different from the risk-free rate. However, if one wants to treat this fact directly, an explicit modeling of the associated risks is required. Here, we use the risk-free rate as net return/cost after hedging these risks. As we shall see, the final formula does not require any knowledge of the risk-free rate, and hence there is no need of its estimation, which is crucial for the practical implementation.

Now, we explain the derivation of the pricing formula. If we denote the present value of the derivative at time \( t \) by \( h^{(i)}(t) \) in terms of currency \((i)\), collateral amount of currency \((j)\) posted from the counterparty is given by

\[
\frac{h^{(i)}(t)}{f^{(i,j)}(t)},
\]

(2.1.2)

where \( f^{(i,j)}(t) \) is the foreign exchange rate at time \( t \) representing the price of the unit amount of currency \((j)\) in terms of currency \((i)\). It should be interpreted that the investor posts the collateral to the counterparty when \( h^{(i)}(t) \) is negative.

These considerations lead to the following calculation for the collateralized derivative price,

\[
h^{(i)}(t) = E_t Q^{(i)} \left[ e^{-\int_t^T r^{(i)}(s) ds} h^{(i)}(T) \right] + f^{(i,j)}(t) E_t Q^{(j)} \left[ \int_t^T e^{-\int_t^s r^{(j)}(u) du} y^{(j)}(s) \left( \frac{h^{(i)}(s)}{f^{(i,j)}(s)} \right) ds \right]
\]

(2.1.3)

where \( E_t Q^{(i)} [\cdot] \) is the time \( t \) conditional expectation under the risk-neutral measure of currency \((i)\), where the money-market account of currency \((i)\)

\[
\beta^{(i)} = \exp \left( \int_0^t r^{(i)}(s) ds \right)
\]

(2.1.4)

is used as the numeraire. Here, the second line of (2.1.3) represents the effective dividend yield from the collateral account, or the cost of posting collateral to the counterparty. One can see the second term changes its sign properly according to the value of the contract.

By changing the measure using the Radon-Nikodym density

\[
\frac{dQ^{(i)}}{dQ^{(j)}} \bigg|_t = \frac{\beta^{(i)} f^{(i,j)}(0)}{\beta^{(j)} f^{(i,j)}(t)}
\]

(2.1.5)

one can show

\[
h^{(i)}(t) = E_t Q^{(i)} \left[ e^{-\int_t^T r^{(i)}(s) ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u) du} y^{(j)}(s) h^{(i)}(s) ds \right].
\]

(2.1.6)

Thus, it is easy to check that the process \( X = \{X_t; \ t \geq 0\} \)

\[
X(t) := e^{-\int_t^0 r^{(i)}(s) ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u) du} y^{(j)}(s) h^{(i)}(s) ds
\]

(2.1.7)
is a $Q^{(i)}$-martingale under the appropriate integrability conditions. This tells us that the process of the option price can be written as

$$dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t) dt + dM(t)$$

with some $Q^{(i)}$-martingale $M$. As a result, we have the following theorem:

**Theorem 1** Suppose that $h^{(i)}(T)$ is a derivative’s payoff at time $T$ in terms of currency $(i)$ and that currency $(j)$ is used as the collateral for the contract. Then, the value of the derivative at time $t$, $h^{(i)}(t)$ is given by

$$h^{(i)}(t) = E_t^{Q^{(i)}} \left[ e^{-\int_t^T r^{(i)}(s) ds} \left(e^{\int_t^T y^{(j)}(s) ds} \right) h^{(i)}(T) \right]$$

$$= D^{(i)}(t, T) E_t^{Q^{(i)}} \left[ e^{-\int_t^T y^{(i,j)}(s) ds} h^{(i)}(T) \right] ,$$

where

$$y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s)$$

with $y^{(i)}(s) = r^{(i)}(s) - c^{(i)}(s)$ and $y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$. Here, we have defined the collateralized zero-coupon bond of currency $(i)$ as

$$D^{(i)}(t, T) = E_t^{Q^{(i)}} \left[ e^{-\int_t^T c^{(i)}(s) ds} \right] .$$

We have also defined the “collateralized forward measure” $T^{(i)}$ of currency $(i)$, for which $E_t^{T^{(i)}} [\cdot]$ denotes the time $t$ conditional expectation. Here, $D^{(i)}(t, T)$ is used as its numeraire, and the associated Radon-Nikodym density is defined by

$$\frac{dT^{(i)}}{dQ^{(i)}} \bigg|_t = \frac{D^{(i)}(t, T)}{\beta^{(i)}(t) D^{(i)}(0, T)}$$

where

$$\beta^{(i)}(t) := \exp \left( \int_0^t c^{(i)}(s) ds \right) .$$

Notice that one has to compensate the dividend yield $y^{(i)}$ of the collateralized zero coupon bond $D^{(i)}$ to make the ratio of numeraire a martingale. This adjustment has changed $\beta^{(i)}$ to $\beta^{(i)}$ in the expression of the Radon-Nikodym density. Since $y^{(i)}$ and $y^{(j)}$ denote the collateral funding cost of currency $(i)$ and $(j)$ respectively, $y^{(i,j)}$ represents the difference of the funding cost between the two currencies. From the last discussion in Introduction, one can expect that this spread plays an important role to calibrate the cross currency basis spreads.

As a corollary of the theorem, we have

$$h^{(i)}(t) = E_t^{Q^{(i)}} \left[ e^{-\int_t^T c^{(i)}(s) ds} h^{(i)}(T) \right] = D^{(i)}(t, T) E_t^{Q^{(i)}} [h^{(i)}(T)]$$

when both of the payment and collateralization are done in a common currency $(i)$. Piterbarg (2010) [25] have derived the consistent formula with (2.1.15) using a partial differential equation (PDE) under a simple Black-Scholes market.
Theorem 1 provides the generic pricing formula under the full collateralization including the case of foreign collateral currency. The result will be used repeatedly throughout the chapter as the foundation of theoretical discussions. We would like to emphasize the importance of the correct understanding for the reason why the collateral rate appears in the discounting. It is the well-known fact that the "effective" discounting rate of an asset with dividend yield \( y \) is given by \( (r - y) \), or the difference of the risk-free rate and the dividend yield. Under the full-collateralization, we can interpret the return (which can be negative) from the collateral account \( y = r - c \) as the dividend yield, which leads to \( r - (r - c) = c \) as the discounting rate. The arguments can be easily extended to the case of foreign collateral currency. Now, it is clear that the so called "OIS-discounting" can be justified only when the collateral rate "\( c \)" is given by the overnight rate of the domestic currency. For example, suppose that there is a trade where the collateral rate or "fee" on the posted collateral, specified in the contractual agreement, is given by the LIBOR, then one should use the LIBOR as the discounting rate of the contract, no matter how different it is from the risk-free rate.

2.2 Term structure modeling

In this section, we discuss the way to give no-arbitrage dynamics to the relevant quantities using Heath-Jarrow-Morton framework. The remaining part of this chapter is mainly based on Fujii, Shimada & Takahashi (2011) [11] and Fujii & Takahashi (2011) [12]. As the early work for the simulation method with multiple curves, see Mercurio (2009) [24], for example.

2.2.1 Dynamics of the forward collateral rate

Firstly, we consider the dynamics of the forward collateral rate, which is defined by

\[
c(t, T) = -\frac{\partial}{\partial T} \ln D(t, T)
\]

or equivalently

\[
D(t, T) = \exp \left( -\int_t^T c(t, s) ds \right).
\]

We also have \( c(t, t) = c(t) \). Suppose that the dynamics of the forward collateral rate under the measure \( Q \) is given by

\[
dc(t, s) = \alpha(t, s) dt + \sigma_c(t, s) \cdot dW^Q_t,
\]

where \( W^Q \) is a \( d \)-dimensional \( Q \)-Brownian motion, \( \alpha, \sigma \in \mathbb{R} \) and \( \alpha, \sigma \in \mathbb{R}^d \) are appropriate adapted processes. Here, we have used the abbreviation

\[
\sigma_c(t, s) \cdot dW^Q_t := \sum_{k=1}^d \left( \sigma_{k}^{(i)}(t, s) \right) dW^Q_{k}(t)
\]

to lighten the notation.
Simple application of Itô’s formula yields
\[
dD^{(i)}(t, T)/D^{(i)}(t, T) = \left\{ c^{(i)}(t) - \int_t^T \alpha^{(i)}(t, s)ds + \frac{1}{2} \left\| \int_t^T \sigma^{(i)}_c(t, s)ds \right\|^2 \right\} dt \\
- \left( \int_t^T \sigma^{(i)}_c(t, s)ds \right) \cdot dW^Q_t. \tag{2.2.5}
\]

From the definition of the zero coupon bond \(D^{(i)}\) in Theorem 1, its drift in \(Q^{(i)}\) must be equal to \(c^{(i)}(\cdot)\). This requires
\[
\alpha^{(i)}(t, s) = \sigma^{(i)}_c(t, s) \cdot \left( \int_t^s \sigma^{(i)}_c(t, u)du \right), \tag{2.2.6}
\]
and as a result, we have
\[
dc^{(i)}(t, s) = \sigma^{(i)}_c(t, s) \cdot \left( \int_t^s \sigma^{(i)}_c(t, u)du \right) dt + \sigma^{(i)}_c(t, s) \cdot dW^Q_t. \tag{2.2.7}
\]

### 2.2.2 Dynamics of the forward LIBOR-OIS spread

We denote the LIBOR of currency \((i)\) fixed at \(T_{n-1}\) and maturing at \(T_n\) as \(L^{(i)}(T_{n-1}, T_n)\). Instead of modeling \(L^{(i)}\) directly, we consider the dynamics of LIBOR-OIS spread, which is defined by
\[
B^{(i)}(T_{n-1}, T_n) = L^{(i)}(T_{n-1}, T_n) - \frac{1}{\delta^{(i)}_n} \left( \frac{1}{D^{(i)}(T_{n-1}, T_n)} - 1 \right) \tag{2.2.8}
\]
where \(\delta^{(i)}_n\) denotes the day-count fraction of \(L^{(i)}\) for the period of \([T_{n-1}, T_n]\). Let us define the forward LIBOR-OIS spread as
\[
B^{(i)}(t; T_{n-1}, T_n) = E_t^{T^{(i)}_n} \left[ B^{(i)}(T_{n-1}, T_n) \right] \\
= E_t^{T^{(i)}_n} \left[ L^{(i)}(T_{n-1}, T_n) \right] - \frac{1}{\delta^{(i)}_n} \left( \frac{D^{(i)}(t, T_{n-1})}{D^{(i)}(t, T_n)} - 1 \right) \tag{2.2.9}
\]
where the Radon-Nikodym density for the forward measure is given by
\[
\frac{d\mathcal{T}^{(i)}_n}{dQ^{(i)}} \bigg|_t = \frac{D^{(i)}(t, T_n)}{\beta^{(i)}_c(t)D^{(i)}(0, T_n)}. \tag{2.2.10}
\]

By construction, \(B^{(i)}(\cdot; T_{n-1}, T_n)\) is a martingale under the forward measure \(\mathcal{T}^{(i)}_n\). Thus, in general, one can write its dynamics as
\[
dB^{(i)}(t; T_{n-1}, T_n)/B^{(i)}(t; T_{n-1}, T_n) = \sigma^{(i)}_B(t; T_{n-1}, T_n) \cdot dW^{T^{(i)}_n}_t \tag{2.2.11}
\]
where \(\sigma^{(i)}_B(\cdot; T_{n-1}, T_n) \in \mathbb{R}^d\) is some adapted process, and \(W^{T^{(i)}_n}_t\) is a \(d\)-dimensional \(\mathcal{T}^{(i)}_n\)-Brownian motion. By Maruyama-Girsanov theorem, one can check that the relation
\[
dW^{T^{(i)}_n}_t = dW^Q_t + \left( \int_t^T \sigma^{(i)}_c(t, s)ds \right) dt \tag{2.2.12}
\]
holds. As a result, one obtains the dynamics under the measure $Q^{(i)}$ as
\[
d B^{(i)}(t; T_{n-1}, T_n) / B^{(i)}(t; T_{n-1}, T_n) = \sigma_B^{(i)}(t; T_{n-1}, T_n) \cdot \left( \int_t^{T_n} \sigma_c^{(i)}(t, s) ds \right) dt + \sigma_B^{(i)}(t; T_{n-1}, T_n) \cdot dW_t^{Q^{(i)}}.
\]
(2.2.13)

2.2.3 Dynamics of the forward currency funding spread

The remaining important ingredient for the term structure modeling is the dynamics of $y^{(i,j)}$. Let us define its instantaneous forward as
\[
y^{(i,j)}(t, T) = -\frac{\partial}{\partial T} \ln Y^{(i,j)}(t, T).
\]
(2.2.14)

Here, we have defined $Y^{(i,j)}$ by
\[
Y^{(i,j)}(t, T) = E_{T}^{(i)} \left[ e^{-\int_t^{T} y^{(i,j)}(t, s) ds} \right]
\]
(2.2.15)

and hence we have
\[
Y^{(i,j)}(t, T) = \exp \left( -\int_t^{T} y^{(i,j)}(t, s) ds \right).
\]
(2.2.16)

Note that the definition is slightly different from that given in [12]. By the same reasoning applied for the forward collateral rate, we can write its dynamics as
\[
d y^{(i,j)}(t, T) = \sigma_y^{(i,j)}(t, T) \cdot \left( \int_t^{T} \sigma_y^{(i,j)}(t, u) du \right) dt + \sigma_y^{(i,j)}(t, T) \cdot dW_t^{T^{(i)}}
\]
(2.2.17)

where $\sigma_y^{(i,j)}(\cdot, T) \in \mathbb{R}^d$ is some appropriate adapted process. Here, we have applied Itô’s formula to $Y^{(i,j)}(t, T)$ and imposing its drift under the forward measure $T^{(i)}$ to be $y^{(i,j)}$, which is implied from (2.2.15). Then, Maruyama-Girsanov theorem gives
\[
d y^{(i,j)}(t, T) = \sigma_y^{(i,j)}(t, T) \cdot \left( \int_t^{T} \left( \sigma_y^{(i,j)}(t, u) + \sigma_c^{(i)}(t, u) \right) du \right) dt + \sigma_y^{(i,j)}(t, T) \cdot dW_t^{Q^{(i)}}.
\]
(2.2.18)

under the standard money-market measure $Q^{(i)}$.

2.2.4 Dynamics of the spot foreign exchange rate

Now, the last piece for the term structure modeling is the dynamics of the spot foreign exchange rate $f_x^{(i,j)}$. Since we have the relation
\[
r_t^{(i)} - r_t^{(j)} = c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)}
\]
(2.2.19)

it is easy to see that the relevant dynamics is given by
\[
d f_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left( c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)} \right) dt + \sigma_{X}^{(i,j)}(t) \cdot dW_t^{Q^{(i)}}.
\]
(2.2.20)
where $\sigma^{(i,j)}_X(\cdot) \in \mathbb{R}^d$ is an appropriate adapted process for the spot FX volatility.

The Radon-Nikodym density between two money-market measures with different currencies are given by

$$ \frac{dQ^{(j)}}{dQ^{(i)}} \bigg|_t = \frac{\beta^{(j)}(t)f^{(i,j)}_x(t)}{\beta^{(i)}(t)f^{(i,j)}_x(0)} = \frac{\beta^{(j)}_c(t)f^{(i,j)}_x(t)}{\beta^{(i)}_c(t)f^{(i,j)}_x(0)} $$

(2.2.21)

where $\beta^{(i,j)}_y(t) = e^{\int_0^t y^{(i,j)}_s ds}$. Since we have the relation

$$ dW^{Q^{(j)}}_t = dW^{Q^{(i)}}_t - \sigma^{(i,j)}_X(t)dt $$

(2.2.22)

it is easy to change the currency measure. For example, one can check that the dynamics of the forward collateral rate of currency $(j)$ becomes

$$ dc^{(j)}(t,s) = \sigma^{(i,j)}_c(t,s) \cdot \left( \int_t^s \sigma^{(i,j)}_c(t,u)du \right) dt + \sigma^{(i,j)}_c(t,s) \cdot dW^{Q^{(i)}}_t $$

(2.2.23)

under the money-market measure of currency $(i)$.

### 2.2.5 Summary of term structure dynamics

In Sections 2.2.1 to 2.2.4, we have derived the dynamics of all the relevant processes in the term structure modeling. For the convenience of readers, we shall summarize the resultant stochastic differential equations (SDEs), which are required for analytical investigation and numerical simulation. Here, we set currency $(i)$ as the base currency.

#### Rates for the base currency

$$ dc^{(i)}(t,s) = \sigma^{(i)}_c(t,s) \cdot \left( \int_t^s \sigma^{(i)}_c(t,u)du \right) dt + \sigma^{(i)}_c(t,s) \cdot dW^{Q^{(i)}}_t $$

(2.2.24)

$$ \frac{dB^{(i)}(t; T_{n-1}, T_n)}{B^{(i)}(t; T_{n-1}, T_n)} = \sigma^{(i)}_B(t; T_{n-1}, T_n) \cdot \left( \int_t^{T_n} \sigma^{(i)}_c(t, s)ds \right) dt + \sigma^{(i)}_B(t; T_{n-1}, T_n) \cdot dW^{Q^{(i)}}_t $$

(2.2.25)

#### Funding spreads

$$ dy^{(i,j)}(t,T) = \sigma^{(i,j)}_y(t,T) \cdot \left( \int_t^T \left( \sigma^{(i,j)}_c(t,u) + \sigma^{(i)}_c(t,u) \right)du \right) dt + \sigma^{(i,j)}_y(t,T) \cdot dW^{Q^{(i)}}_t $$

(2.2.26)

#### Foreign exchange rate

$$ df^{(i,j)}_x(t)/f^{(i,j)}_x(t) = \left( c^{(i)}_t - c^{(j)}_t + y^{(i,j)}_t \right) dt + \sigma^{(i,j)}_X(t) \cdot dW^{Q^{(i)}}_t $$

(2.2.27)

#### Rates for the foreign currencies

$$ dc^{(j)}(t,s) = \sigma^{(j)}_c(t,s) \cdot \left( \int_t^s \sigma^{(j)}_c(t,u)du \right) dt + \sigma^{(j)}_c(t,s) \cdot dW^{Q^{(j)}}_t $$

(2.2.28)

$$ \frac{dB^{(j)}(t; T_{n-1}, T_n)}{B^{(j)}(t; T_{n-1}, T_n)} = \sigma^{(j)}_B(t; T_{n-1}, T_n) \cdot \left[ \left( \int_t^{T_n} \sigma^{(i)}_c(t, s)ds \right) - \sigma^{(i,j)}_c(t) \right] dt + \sigma^{(j)}_B(t; T_{n-1}, T_n) \cdot dW^{Q^{(i)}}_t $$

(2.2.29)
2.3 Curve construction in a single currency

In this section, we explain calibration procedures for swaps in a single currency market, where there exist three different types of swap, which are overnight index swap (OIS), standard interest rate swap (IRS), and tenor swap\(^1\) (TS). The result of the calibration provides the initial condition for the forwards in the previous section. Throughout this section, we assume that the relevant swap is collateralized by the domestic (or payment) currency. Thus, we omit the superscript specifying the currency type.

2.3.1 Overnight Index Swap

As we have seen in the previous section, it is critical to determine the forward curve of overnight rate for the pricing of collateralized contracts. The product called ”overnight index swap” (OIS), which exchanges a fixed coupon and a compounded overnight rate, is particularly useful for this purpose. Here, let us assume that the OIS itself is continuously collateralized by the domestic currency. In this case, using (2.1.15), we get a market condition of

\[
\text{OIS}_N \sum_{n=1}^{N} \Delta_n E^Q \left[ e^{- \int_{0}^{T_n} c(s) ds} \right] = \sum_{n=1}^{N} E^Q \left[ e^{- \int_{0}^{T_n} c(s) ds} \left( e^{\int_{T_{n-1}}^{T_n} c(s) ds} - 1 \right) \right].
\] (2.3.1)

Here, OIS\(_N\) is the market quote of par rate for the length-\(N\) OIS, and \(c(t)\) is the overnight (and hence collateral) rate at time \(t\) of the domestic currency. \(\Delta_n\) denotes a day-count fraction for the fixed leg for a period of \([T_{n-1}, T_n]\). Note that we have converted the overnight rate, which has the convention of daily compounding, to that in continuous compounding for simpler expression. We can simplify the above equation into the form

\[
\text{OIS}_N \sum_{n=1}^{N} \Delta_n D(0, T_n) = D(0, T_0) - D(0, T_N)
\] (2.3.2)

by using collateralized zero-coupon bonds. It is then straightforward to use some appropriate spline-technique to extract smooth curve of \(\{D(0, \cdot)\}\) and hence the forward collateral rate \(\{c(0, \cdot)\}\), too.

2.3.2 Interest rate swap

In a standard interest rate swap (IRS), two parties exchange a fixed coupon and LIBOR for a certain period with a given frequency. For a \(T_0\)-start \(T_N\)-maturing IRS, we have

\[
\text{IRS}_M \sum_{m=1}^{M} \Delta_m D(0, T_m) = \sum_{m=1}^{M} \delta_m D(0, T_m) E^{T_m} [L(T_{m-1}, T_m)]
\] (2.3.3)

as a market condition. Here, IRS\(_M\) is the market IRS quote. Since we already know the term structure of \(D(0, \cdot)\), it is straightforward to extract the set of forward LIBORs \(E^{T_m} [L(T_{m-1}, T_m)]\) for each \(T_m\) from Eq. (2.3.3). By subtracting the forward OIS rate for the same period, we also obtain the set of \(B(0; T_{m-1}, T_m)\).

\(^1\)It is also called money-market basis swap.
2.3.3 Tenor swap

A tenor swap is a floating-vs-floating swap where the parties exchange two LIBORs with different tenors with a fixed spread on the short side, which we call TS basis spread in this article. For example, in a 3m/6m tenor swap, quarterly payments with 3m LIBOR plus spread are exchanged by semi-annual payments of 6m LIBOR flat. The market condition that the tenor spread should satisfy is

\[
\sum_{n=1}^{N} \delta_n D(0, T_n) \left( E^{T_n} [L(T_{n-1}, T_n)] + TS_N \right) = \sum_{m=1}^{M} \delta_m D(0, T_m) E^{T_m} [L(T_{m-1}, T_m)],
\]

(2.3.4)

where \(T_N = T_M\), "m" and "n" distinguish the difference of payment frequency as well as its tenor. TS_N denotes the market quote of the basis spread for the \(T_0\)-start \(T_N\)-maturing tenor swap. Here, \(N > M\) is assumed and hence the spread is added on the short tenor leg. From the above relation and the results obtained in the previous two products, one can extract forward LIBORs with various tenors.

Here, we have explained using slightly simplified terms of contract. In the actual market, it is also common that the coupons of the leg with the short tenor are compounded by LIBOR flat and being paid with the same frequency of the other leg. However, the correction from this compounding is typically very small.

**Remark**

In some market, LIBOR-OIS basis swap is more liquid than the standard OIS swap. If one wants to use this instrument, a simultaneous calibration of LIBOR-OIS and IRS is needed.

2.3.4 Calibration Example

In Figure 2.1, we have given examples of calibrated yield curves for USD market on 2009/3/3 and 2010/3/16, where \(R_{OIS}, R_{3m}\) and \(R_{6m}\) denote the zero rates for OIS (Fed-Fund rate), 3m and 6m forward LIBOR, respectively. \(R_{OIS}(\cdot)\) is defined as \(R_{OIS}(T) = -\ln(D(0, T))/T\). For the forward LIBOR, the "zero-rate curve" \(R_r(\cdot)\) is defined recursively by the relation

\[
E^{T_m} [L(T_{m-1}, T_m; \tau)] = \frac{1}{\delta_m} \left( e^{-R_r(T_{m-1}) T_{m-1}/e^{-R_r(T_m) T_m}} - 1 \right). \tag{2.3.5}
\]

In this section, we have discussed the pricing of various types of interest rate swaps as well as their use for the calibration of the initial term structures of OIS and the forward LIBORs with various tenors. The remaining term structure we need to extract is the one for the funding spread \(y^{(i,j)}\). For this purpose, the understanding of collateralized FX contracts is essential. We study this point in the next section.
2.4 Collateralized FX

Before going to the details of cross currency swaps for curve construction, we need to discuss a collateralized FX forward. We also briefly mention collateralized FX options for completeness.

2.4.1 Collateralized FX forward

Let first consider a collateralized FX forward contract between currency \((i)\) and \((j)\), in which a unit amount of currency \((j)\) is going to be exchanged by \(K\) units amount of currency \((i)\) at the maturity \(T\). The amount of \(K\) is fixed at \(t\), the trade inception time. Assume the contract is fully collateralized by currency \((k)\). Here, the amount of \(K\) that makes this exchange have the present value 0 at time \(t\) is called the forward FX rate.

The break-even condition for the amount of \(K\) becomes

\[
KE^Q_{(i)} \left[ e^{-\int_t^T (c^{(i)} + y^{(i,k)}(s)) ds} \right] = f^{(i,j)}(t) E^Q_{(j)} \left[ e^{-\int_t^T (c^{(j)} + y^{(j,k)}(s)) ds} \right].
\]  

(2.4.1)

Here, the left-hand side represents the value of \(K\) units amount of currency \((i)\) paid at \(T\) with collateralization in currency \((k)\). In the right-hand side the value of a unit amount of currency \((j)\) is represented in terms of currency \((i)\) by multiplying the spot FX rate. By solving the equation for \(K\), we obtain the forward FX as

\[
f^{(i,j)}(t; T; (k)) = f^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)} \left( \frac{Y^{(j,k)}(t, T)}{Y^{(i,k)}(t, T)} \right)
\]  

(2.4.2)

where the last argument denotes the type of collateral currency. In general, the currency triangle relation among forward FXs only holds with the common collateral currency:

\[
f^{(i,j)}(t; T; (k)) \times f^{(j,l)}(t; T; (k)) = f^{(i,l)}(t; T; (k)).
\]  

(2.4.3)
Suppose the same contract is made with a collateral currency either \((i)\) or \((j)\), which seems more natural in the market. In this case, we have one-to-one relation between the forward FX value and the forward funding spread. For example, if currency \((i)\) is used as the collateral, we have

\[
f_x^{(i,j)}(t; T, (i)) = f_x^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)} Y^{(j,i)}(t, T) .
\] (2.4.4)

Therefore, if we can observe the forward FX with various maturities in the market, we can bootstrap the forward curve of \(\{y_t^{(j,i)}(t, \cdot)\}\) since we already know \(D^{(i)}\) and \(D^{(j)}\) from each OIS market. When collateralization is done by currency \((j)\), we can extract \(\{y_t^{(i,j)}(t, \cdot)\}\) by the same arguments.

When the collateral currency is \((i)\), one can easily show that

\[
f_x^{(i,j)}(t) E^Q_t \left[ e^{-\int_t^T (c_x^{(i)} + y_x^{(j,i)}) ds} \right] = E^Q_t \left[ e^{-\int_t^T c_x^{(i)} ds} f_x^{(i,j)}(T) \right] \]

by using the Radon-Nikodym density (2.2.21). Thus, we see

\[
f_x^{(i,j)}(t; T, (i)) = E^T_t \left[ f_x^{(i,j)}(T) \right] \]

and hence the forward FX collateralized by its domestic currency is a martingale under the associated forward measure. In more general situation, we have the expression

\[
f_x^{(i,j)}(t; T, (k)) = E^T_t \left[ f_x^{(i,k)}(T) \right] \]

where the new forward measure is defined by

\[
\frac{dT^{(i,k)}}{dQ^{(i)}} \bigg|_t = \frac{D^{(i)}(t, T) Y^{(i,k)}(t, T)}{\beta^{(i)}(t) \beta^{(i,k)}(t) D^{(i)}(0, T) Y^{(i,k)}(0, T)} .
\] (2.4.8)

One can see that the zero coupon bond of currency \((i)\) collateralized by currency \((k)\)

\[
E^Q_t \left[ e^{-\int_t^T (c_x^{(i)} + y_x^{(i,k)}) ds} \right] = D^{(i)}(t, T) Y^{(i,k)}(t, T)
\]

is associated as a numeraire for the forward measure \(T^{(i,k)}\).

### 2.4.2 FX option

Just for completeness, we give a brief remark on collateralized FX European options. Let us consider a \(T\)-maturing European call option on \(f_x^{(i,j)}\) with strike \(K\) collateralized by a currency \((k)\). The present value of the option is

\[
PV_t = E^Q_t \left[ e^{-\int_t^T (c_x^{(i)} + y_x^{(i,k)}) ds} \left( f_x^{(i,j)}(T) - K \right)^+ \right] ,
\]

where \(X^+\) denotes \(\max(X, 0)\). Since we can rewrite it as

\[
PV_t = D^{(i)}(t, T) Y^{(i,k)}(t, T) E^T_t \left[ \left( f_x^{(i,j)}(T; T, (k)) - K \right)^+ \right] \]

(2.4.11)
it is enough to know the dynamics of $f_x(t; T, (k))$ under the measure $T^{(i,k)}$. By applying Itô formula, we obtain
\[
df^{(i,j)}(t; T, (k)) / f^{(i,j)}(t; T, (k)) = \left\{ \sigma^{(i,j)}(t) + \Gamma^{(i,k)}(t, T) - \Gamma^{(j,k)}(t, T) \right\} \cdot dW^T_{(i,k)}(t, 2.4.12)
\]
where $W^T_{(i,k)}$ is a $d$-dimensional $T^{(i,k)}$-Brownian motion. Here, for $m = \{i, j\}$, we have defined
\[
\Gamma^{(m,k)}(t, T) = \int_t^T \left( \sigma^{(m)}_c(t, s) + \sigma^{(m,k)}_y(t, s) \right) ds .
\] (2.4.13)

Although there is no closed-form solution except when all the volatility functions are deterministic, asymptotic expansion technique can be applied to derive analytical approximate solution. See, for example, [29, 30] and references therein.

In Section 2.4, we have derived the formula for the collateralized FX forward, its martingale property under the associated forward measure as well as the pricing formula for the collateralized FX options. They are indispensable tools for those who want to handle the situation involving multiple currencies.

### 2.5 Curve construction in multiple currencies

In this section, we finally proceed to discuss the curve construction in a multi-currency environment. For the calibration of forward collateral rate and LIBOR-OIS spreads of each currency can be done in exactly the same way as in the previous sections. The missing component is the forward currency funding spread $\{y^{(i,j)}(0, \cdot)\}$.

#### 2.5.1 Cross currency swaps

Although the pricing of FX forward is simple, its liquidity is limited only for short maturities. In this section, we discuss how to make the term structure consistent with cross currency swap (CCS) [12], which is more liquid across medium to long terms. Contracts with USD crosses are the most popular in the market, where 3m USD LIBOR flat is exchanged with 3m LIBOR of non-USD currency with additional basis spread. There are two types of CCS, one is Constant Notional CCS (CNCCS), and the other is Mark-to-Market CCS (MtMCCS). In a CNCCS, the notional of both legs are fixed at the inception of the trade and kept constant until it expires. However, in a MtMCCS, the notional of USD is adjusted to that of the non-USD currency at the every LIBOR fixing time using the spot FX rate. Although the required calculation becomes slightly more complicated, it is desirable to use MtMCCS for calibration due to its better liquidity.

#### Mark-to-Market cross currency swap

We consider a spot-start $T_N$-maturing MtMCCS with a currency pair $(i, j)$, where the leg of currency $(i)$ (intended to be USD) needs notional refreshments. We assume that the collateral is posted in currency $(i)$, which seems popular in the market. This is expected to be a standard convention in the forthcoming standard credit support annex (SCSA),
too. We assume that the notional of currency (j) is equal to 1, and use the convention $T_0 = 0$ in the following.

The present value of the cash flow of currency (j) is given by

$$PV_{MtM}^{(j)} = -E^{Q(j)}\left[e^{-\int_0^T (c_{j;i}^n + y_{j;i}^n)ds}\right] + E^{Q(j)}\left[e^{-\int_0^T (c_{j;i}^n + y_{j;i}^n)ds}\left(L^{(j)}(T_{n-1}, T_n) + X_{M}^{MtM}\right)\right]$$


(2.5.1)

where $X_{M}^{MtM}$ is the market basis spread for the length-N MtMCCS. It can be calculated as

$$PV_{MtM}^{(j)} = -1 + D^{(j)}(0, T_N)Y^{(j,i)}(0, T_N)$$

$$+ \sum_{n=1}^N \delta_n^{(j)} D^{(j)}(0, T_n)Y^{(j,i)}(0, T_n)\left(E^{T_n^{(j,i)}}\left[L^{(j)}(T_{n-1}, T_n)\right] + X_{M}^{MtM}\right)$$

(2.5.2)

where $E^{T_n^{(j,i)}}[\cdot]$ is defined as (2.4.8).

If there exists a liquid IRS market of currency (j) but collateralized with a foreign currency (i), we can extract

$$E^{T_n^{(j,i)}}\left[L^{(j)}(T_{n-1}, T_n)\right] = E^{T_n^{(j)}}\left[L^{(j)}(T_{n-1}, T_n)\right].$$ (2.5.3)

as we have done in Section 2.3.2. Unfortunately though, this is typically not the case. Therefore, let us approximate it by

$$E^{T_n^{(j,i)}}\left[L^{(j)}(T_{n-1}, T_n)\right] = E^{T_n^{(j)}}\left[L^{(j)}(T_{n-1}, T_n)\right].$$ (2.5.4)

Let us explain the conditions under which the above approximation works. It is clear that we have the following backward SDEs,

$$E^{T_n^{(j)}}\left[L^{(j)}(T_{n-1}, T_n)\right] = L^{(j)}(T_{n-1}, T_n) - \int_t^{T_n} \sigma_{L^{(j)}}(s) \cdot dW_t^{T_n^{(j)}}$$ (2.5.5)

$$E^{T_n^{(j)}}\left[L^{(j)}(T_{n-1}, T_n)\right] = L^{(j)}(T_{n-1}, T_n) - \int_t^{T_n} \sigma_{L^{(j)}}(s) \cdot dW_t^{T_n^{(j)}}$$ (2.5.6)

where $\sigma_{L^{(j)}}(\cdot)$ is an appropriate adapted process for the LIBOR volatility. Since we know, from the Maruyama-Girsanov’s theorem that

$$dW_t^{T_n^{(j,i)}} = dW_t^{T_n^{(j)}} + \left(\int_t^{T_n} \sigma_{y^{(j,i)}}(t, s)ds\right)dt$$ (2.5.7)

the equality in Eq.(2.5.4) holds when

$$\sigma_{L^{(j)}}(s) \cdot \left(\int_s^{T_n} \sigma_{y^{(j,i)}}(s, u)du\right) = 0, \text{ a.e. } s \in [0, T_n]$$ (2.5.8)

i.e., zero quadratic covariance between $L^{(j)}$ and $y^{(j,i)}$. Practically, if the covariance is small enough, Eq.(2.5.4) still holds approximately. Under this condition, $\{Y^{(j,i)}(0, T_n)\}_n$ are only unknown in $PV_{MtM}^{(j)}$. 

22
On the other hand, the present value of the cash flow of currency \((i)\) is given by

\[
PV_{\text{MtM}}^{(i)} = -\sum_{n=1}^{N} E^{Q(i)} \left[ e^{-\int_{0}^{T_n} \sigma(s) dW_{x}(s)} f^{(i,j)}(T_n-1) \right] \\
+ \sum_{n=1}^{N} E^{Q(i)} \left[ e^{-\int_{0}^{T_n} \sigma(s) dW_{x}(s)} f^{(i,j)}(T_n-1) \left( 1 + \delta^{(i)}(T_n-1, T_n) \right) \right] \\
= \sum_{n=1}^{N} \delta^{(i)}(0, T_n) E^{Q(i)} \left[ f^{(i,j)}(T_n-1) B^{(i)}(T_n-1, T_n) \right].
\]

(2.5.9)

The last expression is volatility and correlation dependent in general. To make calibration tractable, we assume that the LIBOR-OIS spread and FX rate are independent or their covariance is negligible. In addition, we neglect the small convexity correction arising from the timing mismatch. Both of the assumptions look reasonable in most situations. Then, we can simplify the expression as

\[
PV_{\text{MtM}}^{(i)} \approx \sum_{n=1}^{N} \delta^{(i)}(0, T_n) E^{Q(i)} \left[ f^{(i,j)}(0; T_n-1, (i)) B^{(i)}(0; T_n-1, T_n) \right].
\]

(2.5.10)

By converting its value in terms of currency \((j)\), we have

\[
f^{(j,i)}(0) PV_{\text{MtM}}^{(i)} \approx \sum_{n=1}^{N} \delta^{(i)}(0, T_n) E^{Q(i)} \left[ f^{(j,i)}(0; T_n-1, (i)) B^{(i)}(0; T_n-1, T_n) \right].
\]

(2.5.11)

As a result, under the simplifying assumptions, one can see that the only unknowns in both \(PV_{\text{MtM}}^{(i)}\) and \(PV_{\text{MtM}}^{(j)}\) are the set of \(\{Y^{(j,i)}(0, T_n)\}_n\). Thus, one can extract the forward funding spread \(\{y^{(j,i)}(0, \cdot)\}\) by imposing a market condition

\[
PV_{\text{MtM}}^{(j)} = f^{(j,i)}(0) PV_{\text{MtM}}^{(i)}
\]

(2.5.12)

for all the traded MtMCCSs.

**Remark**

When the simplifying assumptions do not hold, we are forced to do iterative calibration. This is because the calibration of CCS requires all the relevant volatility (and correlation) information, but the calibration of options (particularly FX options) requires the \(y^{(j,i)}\) curve. A possible calibration procedure in this case is:

1. Calibrate MtMCCS by using the simplified assumptions.
2. Calibrate the relevant options using the curves obtained from (1).
3. Calculate MtMCCS using the full volatility information obtained (2) either by Monte Carlo simulation or by some analytic method, and then readjust \(y^{(j,i)}\) curve to make the market condition of each MtMCCS satisfied.
4. Repeat (1) to (3).

Considering the size of corrections, small number of iterations would be enough to achieve a reasonable result.
Constant notional cross currency swap

In a CNCCS, the notional of currency (i) leg is also kept constant. We assume the same setup used for the MtMCCS in the previous section and consider a spot-start $T_N$-maturing CNCCS collateralized by currency (i). The present value of currency (j) leg is calculated in the same way as before:

$$PV_{CN}^{(j)} = -1 + D^{(j)}(0, T_N) Y^{(j,i)}(0, T_N)$$

$$+ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0, T_n) Y^{(j,i)}(0, T_n) \left( E^{T_n}_{T_n^{(j,i)}} \left[ L^{(j)}(T_{n-1}, T_n) \right] + X^{CN}_N \right)$$

where $X^{CN}_N$ is the market basis spread for the length-$N$ CNCCS.

Since the notional of USD (and hence currency (i)) leg is fixed at time 0 without further adjustment, its present value is given by

$$PV_{CN}^{(i)} = f_x^{(i,j)}(0) \left\{ -1 + D^{(i)}(0, T_N) + \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0, T_n) E^{T_n}_{T_n^{(i,j)}} \left[ L^{(i)}(T_{n-1}, T_n) \right] \right\}$$

$$= f_x^{(i,j)}(0) \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0, T_n) E^{T_n}_{T_n^{(i,j)}} \left[ B^{(i)}(T_{n-1}, T_n) \right]$$

and there is no need of approximation for this leg.

Therefore, also from CNCCS, we can extract $\{y^{(j,i)}(0, \cdot)\}$ by imposing market conditions

$$f_x^{(j,i)}(0) PV_{CN}^{(i)} = PV_{CN}^{(j)}$$

(2.5.15)

for available swaps. At the current market, MtMCCS seems a more appropriate calibration instrument due to its higher liquidity, though.

Price difference of the two cross currency swaps

Although the two types of CCS have existed for many years, their price difference has not been recognized properly until recently. One can easily check that

$$PV_{MtM}^{(j)} - PV_{CN}^{(j)} = (X_N^{MtM} - X_N^{CN}) \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0, T_n) Y^{(j,i)}(0, T_n)$$

(2.5.16)

and

$$f_x^{(j,i)}(0) \left( PV_{MtM}^{(i)} - PV_{CN}^{(i)} \right) = \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0, T_n) E^{T_n}_{T_n^{(i,j)}} \left[ \left( \frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} \right) - 1 \right] \left[ B^{(i)}(T_{n-1}, T_n) \right]$$

(2.5.17)

Therefore, we obtain

$$X_N^{MtM} - X_N^{CN} = \frac{\sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0, T_n) E^{T_n}_{T_n^{(i,j)}} \left[ \left( \frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} \right) - 1 \right] \left[ B^{(i)}(T_{n-1}, T_n) \right]}{\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0, T_n) Y^{(j,i)}(0, T_n)}$$

(2.5.18)
which is not zero in general.

For example, let us consider the two USDJPY CCSs. The two swaps should have the same basis spreads if USD LIBOR-OIS spreads are all zero. This held approximately well before the Lehman crisis but the spread has been far from zero since then. If USD interest rate level is higher than JPY, as is usually the case, the formula tells us that the spread for MtMCCS is quite likely to be higher than that of CNCCS, i.e., \((X_{N}^{MtM} - X_{N}^{CN}) > 0\). The size of spread may not be negligible dependent on situations, and hence it is worthwhile paying enough attention to the difference in this post crisis era.

2.5.2 Historical behavior of the funding spread

Now, let us check the historical behavior of \(R_{y}(EUR,USD)\) and \(R_{y}(JPY,USD)\). For the calibration of MtMCCS, we have used the simplifying assumptions stated in Section 2.5.1. Here, the spread \(R_{y}\) is defined as

\[
R_{y}(j,i)(T) = -\frac{1}{T} \ln Y^{(j,i)} = \frac{1}{T} \int_{0}^{T} y^{(j,i)}(0, s) ds .
\]  

In Figure 2.2, we have shown historical behaviors of basis spreads of 5y MtMCCS, corresponding \(R_{y}(X,USD)(5y)\), and difference of \(R_{3m}(5y) - R_{OIS}(5y)\) between the two currency pairs denoted by \(\Delta\)LIBOR-OIS(5y; USD, X) \(^2\). Here, “X” stands for either EUR or JPY. The time series for JPY are limited since we could not get the long-term OIS data from Bloomberg. As one can see from the previous sections, the value of CCS spread has contribution from the LIBOR-OIS and the funding spreads between the two currencies. In Figure 2.2, \(R_{y}(X,USD)(5y) + \Delta\)LIBOR-OIS(5y; USD, X) well agrees with the 5y MtMCCS spread with typical error smaller than a few bps. One can observe that a significant portion of the movement of CCS spreads stems from the change of \(y^{(i,j)}\), rather than the difference of LIBOR-OIS spreads. In fact, the level (difference)-correlation between \(R_{y}\) and CCS spread is quite high, which is about 93% (69%) for EUR and about 70% (48%) for JPY for the historical series used in the figure. Except the CCS basis spread, \(y\) does not seem to have persistent correlations with other market variables such as OIS, IRS and FX forwards.

In Figures 2.3 and 2.4, term structures of calibrated (EUR, USD) and (JPY, USD) funding spread are provided for several dates. Both of them have very similar term structure to the corresponding cross currency swaps.

2.6 OIS discounting and choice of collateral currency

We now consider implications of the full collateralization. It is straightforward to see when payment and collateral currencies are the same. As in Eq. (2.1.15), discounting rate is now determined by collateral (or ON) rate rather than LIBORs. Hence, in the presence of the current level of LIBOR-OIS spread of (~20)bps, the conventional LIBOR discounting method results in significant underestimation of the value of future payments, which can even be a few percentage points for long maturities. Considering the mechanism of collateralization, financial firms need to hedge the move of OIS in addition to LIBORs.

\(^{2}\)It can be interpreted as the difference of LIBOR-OIS spread between USD and X.
In particular, the risk of floating-rate payments needs to be checked carefully, since the overnight rate can move in the opposite direction to LIBOR as was observed in this financial crisis. In Figure 2.5, the present values of LIBOR floating legs with a final principal payment

$$PV = \sum_{n=1}^{N} \delta_n D(0,T_n) E^{T_n} [L(T_{n-1}, T_n; \tau)] + D(0,T_N)$$

are given for various maturities. If the traditional LIBOR discounting is used, the stream of LIBOR payments should have the constant present value "1", which is obviously wrong from our results. This observation is very important in risk-management point of view.
since financial firms may overlook the quite significant interest-rate risk exposure when traditional interest rate models are used.

If a trade with payment currency \( j \) is collateralized by foreign currency \( i \), a multiplicative modification to the discounting factor appears (See Theorem1 with \( h(T) = 1 \)):

\[
e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{(j)} \left[ e^{-\int_t^T y^{(j,i)}(s)ds} \right].
\]  

(2.6.2)

From Figures 2.3 and 2.4, one can see that posting USD as collateral tends to be expensive from the viewpoint of collateral payers, which is particularly the case when everyone is seeking USD cash in illiquid market. Since the funding spread is defined by

\[
y^{(JPY,USD)} = y^{JPY} - y^{USD},
\]  

(2.6.3)

the fact \( y^{(JPY,USD)} < 0 \) seen from the figures is indicating that the cost of USD collateral is higher than that of JPY. For example, from Figure 2.4, one can see that the value of JPY payment in 10 years time is higher by around 3% when it is collateralized by USD instead of JPY. Notice that the collateral payers have negative mark-to-market by definition, and thus higher present value of future cash flow works against them. The same effects tend to be more profound for emerging currencies where the implied CCS basis spread can easily be negative 100 bps or smaller.

### 2.7 Embedded optionality in collateral agreements

In this section, we discuss the embedded cheapest-to-deliver (CTD) options in a collateral agreement. In some cases, financial firms make contracts with CSA allowing several currencies (and/or assets) as eligible collateral. In some jurisdictions, payers of collateral can switch the collateral whenever they want, but in others they are required to obtain consent from their counterparties to do so. In the following, let us assume that a collateral payer can freely switch the collateral within the eligible set.
2.7.1 Symmetric Cases

Let us consider the valuation problem of a contract which pays $h^{(i)}(T)$ at $T$ under the full collateralization with the eligible cash collateral set $C$. Assume that the collateral payer, which can be either the investor or the counterparty, optimally chooses the collateral currency. Since the optimal currency is the one that has the lowest funding cost, it is determined by

$$\min_{j \in C} y^{(j)} = \min_{j \in C} \left( r^{(j)} t - c^{(j)} t \right). \tag{2.7.1}$$

Following the same arguments used for the derivation of Theorem 1, one can show the valuation formula becomes

$$h^{(i)}(t) = E_t^Q \left[ e^{-\int_t^T r^{(i)} s h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)} r s - \min_{j \in C} y^{(j)} s h^{(i)}(s) ds} \right]. \tag{2.7.2}$$

which then yields

$$h^{(i)}(t) = E_t^Q \left[ e^{-\int_t^T r^{(i)} s + \max_{j \in C} y^{(j)} s ds h^{(i)}(T)} \right] = D^{(i)}(t, T) E_t^{T(i)} \left[ e^{-\int_t^T \max_{j \in C} y^{(j)} s ds h^{(i)}(T)} \right]. \tag{2.7.3}$$

Thus, except the correlation effects, the effective discount factor is now given by

$$D^{(i)}(t, T) E_t^{T(i)} \left[ e^{-\int_t^T \max_{j \in C} y^{(j)} s ds} \right]. \tag{2.7.4}$$

One can check that the cheapest collateral leads to the highest effective discounting rate.

As a special example, let us consider a two currency case; $C = \{i, j\}$. In this case, the multiplicative modification factor turns out to be

$$E_t^{T(i)} \left[ e^{-\int_t^T \max\{y^{(i)}(s), 0\} ds} \right]. \tag{2.7.5}$$
In Figure 2.6, we have plotted the modification factor given in Eq. (2.7.5), for \( i = \text{EUR} \) and \( j = \text{USD} \) as of 2010/3/16. We have used Hull-White model for the dynamics of \( y^{(EUR,USD)}(\cdot) \) with a mean reversion parameter 1.5% per annum and the set of volatilities, \( \sigma = 0, 25, 50 \) and 75 bps, respectively. These are annualized volatilities in absolute terms. For simplicity, \( y^{(i,j)} \) is assumed to be independent from \( c^{(i)} \) in the simulation. The 3m-roll historical volatilities of \( y^{(EUR,USD)} \) instantaneous forwards, which are annualized in absolute terms, are given in Figure 2.7. In a calm market, they tend to be 50 bps or so, but they were more than a percentage point just after the market crisis, which is reflecting a significant widening of the CCS basis spread to seek USD cash in illiquid market. The curve labeled by USD (EUR) denotes the modification of the discount factor when only USD (EUR) is eligible collateral for the ease of comparison. One can easily see that there is significant impact when the collateral currency chosen optimally. For example, from Figure 2.6, one can see if the two parties continue to switch the collateral currency between EUR and USD optimally, it increases the effective discounting rate by roughly 50 bps even when the annualized volatility of spread \( y^{(EUR,USD)} \) is 50 bps.

Figure 2.6: Modification of EUR discounting factors based on HW model for \( y^{(EUR,USD)} \) as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

### 2.7.2 Asymmetric Cases

Once understanding the significance of embedded optionality, it is a natural question to ask what happens when a collateral agreement is asymmetric. For example, one party may have multiple choices of eligible collateral but not for the other. This kind of situation may arise due to asymmetric CSA, or even in the case of symmetric CSA if one party does not have an easy access to the cheapest collateral or lacking the sophistication in collateral management. In these situations, the party who has limited choice of collateral may lose significant value.

For concreteness, consider a contract with cumulative dividend \( N^{(i)} = \{ N_t^{(i)}, t \in [0, T] \} \) with currency \( (i) \). Assume that the investor (party 1) can only use the currency \( (i) \) but
the counterparty (party 2) can freely choose the collateral currency from the set $C$ which includes the currency $(i)$ and several others. In this case, the ex-dividend value process $V = \{V_t, t \in [0, T]\}$ is given by the following formula from the viewpoint of the investor:

$$V_t = E_t^Q (\int_\tau^T e^{-\int_t^s r_u^{(i)} du} dN_s^{(i)})$$

$$+ E_t^Q (\int_\tau^T e^{-\int_t^s r_u^{(i)} du} \left( -y_s^{(i)} - V_s \mathbf{1}_{\{V_s < 0\}} + \min_{j \in C} y_s^{(j)} V_s \mathbf{1}_{\{V_s \geq 0\}} \right) ds) \quad (2.7.6)$$

The second line represents the fact that the investor needs to post collateral with cost $y^{(i)} = r^{(i)} - c^{(i)}$ when the mark-to-market is negative, and the counterparty chooses the cheapest collateral currency that only produces $\min_{j \in C} y^{(j)}$ to the investor when the mark-to-market is positive.

The above formula can be rewritten as

$$V_t = E_t^Q \left[ \int_\tau^T \exp \left( -\int_t^s (c_u^{(i)} + \max_{j \in C} y_u^{(i,j)} 1_{\{V_u \geq 0\}}) du \right) dN_s^{(i)} \right] \quad (2.7.7)$$

Since this is a non-linear forward backward stochastic differential equation (FBSDE), it is impossible to solve explicitly. Using Gateaux derivative, one can show that the first order approximation is given by [13]

$$V_t \simeq \nabla_t - E_t^Q (\int_\tau^T e^{-\int_t^s r_u^{(i)} du} \max_{j \in C} y_s^{(i,j)} V_s 1_{\{V_s \geq 0\}} ds) \quad (2.7.8)$$

where

$$\nabla_t = E_t^Q \left[ \int_\tau^T e^{-\int_t^s c_u^{(i)} du} dN_s^{(i)} \right] \quad (2.7.9)$$
is the contract value when the both parties use the currency \((i)\) as collateral. Due to the assumption that \(i \in C\), the expectation in the second term is guaranteed to be positive, which represents the optionality sold to the counterparty.

If the above asymmetry is due to an asymmetric CSA, then the investor can claim the fee to the counterparty in principle. However, it is impossible when the situation arises only due to the lack of sophistication in collateral management of the investor. In this case, the investor may face significant loss by effectively selling CTD options to the counterparty by free of charge. Some numerical examples are available in [13]. One can see that it is best to avoid making flexible collateral agreements if it is incapable of posting the cheapest collateral.

### 2.8 Standard Credit Support Annex and USD Silo

We have just seen that the right of the collateral payer to choose a collateral each time induces CTD option to the contract. The value of this option can be quite significant especially when the CCS market is volatile. Considering the ongoing concern on EUR-zone sovereign debt crisis and the historical level of CCS basis spreads, it is easy to imagine that these complexities evoke disputes on valuation, and hence reducing market transparency and liquidity. It also makes hedging and novation quite difficult when the associated CSA is different.

In order to mitigate the problem, ISDA is planning to release Standard Credit Support Annex (SCSA) [19] in 2012 to seek the standardization of CSA \(^3\). It is argued that there will be one collateral requirement per currency, delivered in each currency (or converted to a single currency with an interest rate adjustment overlay) so that there is no embedded optionality in collateral agreements. In SCSA, each trade will be assigned to one of the 17 DCC (Designated Collateral Currency) Silos of G5 (USD, EUR, GBP, CHF, and JPY) and other 12 currencies. For a trade that involves only a single currency included in the above currency set will be assigned to the corresponding DCC silo and hence collateralized by the same currency. In this case, the discounting rate is simply its OIS rate. If a contracting party chooses different currency to post, the collateral rate will be adjusted in such a way that it gives the same economic effects as the standard case. Those currencies not included in the above set as well as multiple-currency trades are likely to be assigned to the USD silo, where the collateralization is done by USD cash, or a different currency with adjusted collateral rate to achieve the same economic effects.

One possible reason that some currencies of emerging countries are assigned to the USD silo is the lack of liquid domestic OIS markets for them. If SCSA becomes the dominant choice among practitioners, we need to consider the fundamental issues, such as the swap curve construction, interest rates and FX modeling under a peculiar situation, in which most of the trades are collateralized by a foreign currency (i.e. USD), and hence it is impossible to rely on the domestic OIS rate for valuation. In the final part of this chapter, we provide a simple closed framework for the valuation issues in the USD silo. This section is based on Fujii & Takahashi (2011) [14].

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\(^3\)At the time of writing this document, we have not yet seen the final release.
2.8.1 Discounting under the USD silo

In the reminder of this chapter, we fix currency (i) as USD. From Theorem 1, one can see that
\[
r^{(j)} := r^{(j)} - y^{(i)}
\]  
(2.8.1)
is the effective discounting rate of currency (j) in the USD silo. We define the USD collateralized zero coupon bond as
\[
D^{(j)}(t, T) = E^{Q^{(j)}}(t) [e^{-\int_t^T r^{(j)}(s) ds}] .
\]  
(2.8.2)
Note that we have \(D^{(i)}(t, T) = D^{(i)}(t, T)\). It is also useful to consider the USD adjusted money-market account of currency (j) defined by
\[
\beta^{(j)}(t) = \exp\left(\int_0^t r^{(j)}(s) ds\right),
\]  
(2.8.3)
and then it is straightforward to check that \(D^{(j)}(\cdot, T)/\beta^{(j)}(\cdot)\) is a positive \(Q^{(j)}\)-martingale. Thus, we can define the USD-collateralized forward measure \(\overline{T}^{(j)}\) by
\[
d\overline{T}^{(j)} \bigg| dQ^{(j)} \bigg|_t = \frac{D^{(j)}(t, T)}{\beta^{(j)}(t) D^{(j)}(0, T)}. \]  
(2.8.4)
Using these notations, the price of a contract which pays \(h^{(j)}(T)\) in currency (j) at time \(T\) under the USD silo is given by
\[
h^{(j)}(t) = D^{(j)}(t, T) E^{\overline{T}^{(j)}}(t) [h^{(j)}(T)].
\]  
(2.8.5)
Note that the measure \(\overline{T}^{(j)}\) is actually \(T^{(j,i)}\), but we use the new convention in the remaining part of this chapter.

2.8.2 FX forward under the USD silo

FX forward under the USD silo is simply a special case discussed in Section 2.4.1. The FX forward of currency pair \((j, k)\) with maturity \(T\), denoted by \(\overline{f}^{(j,k)}(t; T)\), is given by
\[
\overline{f}^{(j,k)}(t; T) = \overline{f}^{(j,k)}(t; T, (i))
\]  
(2.8.6)
and it satisfies
\[
\overline{f}^{(j,k)}(t; T) = E^{\overline{T}^{(j)}}(t) [f^{(j,k)}(T)].
\]  
(2.8.7)
In particular, the FX forward of (USD, j) pair is
\[
f^{(i,j)}(t; T) = f^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)}. \]  
(2.8.8)
We assume that there exist a Fixed-versus-LIBOR interest rate swap market of currency \((j)\) and a \((j, i)\)-MtMCCS (Mark-to-Market Cross Currency Swap) market. Both types of contracts are assumed to be USD collateralized. For simplicity, we will assume the LIBOR tenors are the same for both of the products. The technique adopted in [10, 12] are readily applicable to eliminate this assumption if necessary.

### 2.9.1 IRS in the USD silo

First, let us consider a standard Fixed-vs-LIBOR swap for currency \((j)\) allocated in the USD silo. At time \(t = 0\), the market quote for \(T_0\)-start \(T_M\)-maturing interest rate swap denoted by \(S_{(j)M}\) should satisfy

\[
S_{(j)M}^j \sum_{m=1}^{M} \Delta_m \overline{D}^{(j)}(0, T_m) = \sum_{m=1}^{M} \delta_m \overline{D}^{(j)}(0, T_m) E_T^{m(M)} [L^{(j)}(T_{m-1}, T_m)] .
\]  

(2.9.1)

Here, we distinguish the day-count fraction of fixed and floating legs by \(\Delta\) and \(\delta\), which are not necessarily the same. Thus the market par rate is expressed as

\[
S_{(j)M}^j = \frac{\sum_{m=1}^{M} \delta_m \overline{D}^{(j)}(0, T_m) E_T^{m(M)} [L^{(j)}(T_{m-1}, T_m)]}{\sum_{m=1}^{M} \Delta_m \overline{D}^{(j)}(0, T_m)}
\]  

(2.9.2)

using the LIBORs and USD-collateralized zero coupon bonds.

### 2.9.2 MtMCCS in the USD silo

We now consider a MtMCCS between currency \((j)\) in the USD silo and USD itself. The calculation is completely parallel with those in Section 2.5.1, except the fact that IRS with USD collateralization is available.

The present value of \((j)\)-leg of a \(T_0\)-start \(T_M\)-maturing MtMCCS is then given by

\[
PV^{(j)} = -\overline{D}^{(j)}(0, T_0) + \overline{D}^{(j)}(0, T_M) + \sum_{m=1}^{M} \delta_m \overline{D}^{(j)}(0, T_m) E_T^{m(M)} [L^{(j)}(T_{m-1}, T_m)] + X_M
\]

(2.9.3)

where \(X_M\) is the basis spread for the period-\(M\) MtMCCS. Using the result of IRS, it can be simplified as

\[
PV^{(j)} = -\overline{D}^{(j)}(0, T_0) + \overline{D}^{(j)}(0, T_M) + \left(\frac{\Delta}{\delta} S_{(j)M}^j + X_M\right) \sum_{m=1}^{M} \delta_m \overline{D}^{(j)}(0, T_m)
\]

(2.9.4)

Here, we have approximated the ratios of day-count fraction \((\Delta_m/\delta_m)\) by the single constant \(\Delta/\delta\) for simplicity.

\[\text{For example, if the floating leg is based on Act360, and the fixed leg Act365, the ratio } \Delta/\delta \text{ is given by } \Delta/\delta = 360/365.\]
For the USD leg, the calculation is completely the same as before

\[
PV^{(i)} = -\sum_{m=1}^{M} E^{Q^{(i)}} \left[ e^{-\int_0^{T_{m-1}} c^{(i)}(s) ds} f^{(i,j)}(T_{m-1}) \right] \\
+ \sum_{m=1}^{M} E^{Q^{(i)}} \left[ e^{-\int_0^{T_m} c^{(i)}(s) ds} f^{(i,j)}(T_{m-1}) \left( 1 + \delta_m^{(i)} L^{(i)}(T_{m-1}, T_m) \right) \right]
\]

which can be rewritten as

\[
PV^{(i)} = \sum_{m=1}^{M} \delta_m^{(i)} D^{(i)}(0, T_m) E^{Q^{(i)}} \left[ f^{(i,j)}(T_{m-1}) B^{(i)}(T_{m-1}, T_m) \right]. \tag{2.9.5}
\]

Using the same assumptions when (2.5.11) is derived, we obtain

\[
f^{(j,i)}(0) PV^{(i)} \simeq \sum_{m=1}^{M} \delta_m^{(j)} D^{(j)}(0, T_m) D^{(i)}(0, T_{m-1}) B^{(i)}(0; T_{m-1}, T_m). \tag{2.9.6}
\]

As a result, we obtain the par CCS basis spread (except the associated approximation errors) as

\[
X_M = \left( S^{(j)}_M - \Delta S^{(j)}_M \right) + \frac{\sum_{m=1}^{M} \delta_m^{(j)} \left( D^{(j)}(0, T_m) B^{(i)}(0; T_{m-1}, T_m) D^{(j)}(0, T_{m-1}) \right)}{\sum_{m=1}^{M} \delta_m^{(j)} D^{(j)}(0, T_m)} \tag{2.9.7}
\]

where

\[
S^{(j)}_M := \frac{D^{(j)}(0, T_0) - D^{(j)}(0, T_M)}{\sum_{m=1}^{M} \delta_m^{(j)} D^{(j)}(0, T_m)} \tag{2.9.8}
\]

denotes the effective swap rate associated with the discounting rate of currency \((j)\).

### 2.9.3 Curve construction in the USD silo

Now, let us discuss the curve construction in the USD silo. As we have mentioned earlier, we are currently assuming that there is no swap markets collateralized by the domestic currency. In this case, we cannot separate the curve of a domestic collateral rate (and hence overnight rate) \(c^{(j)}\) and the curve of funding spreads \(y^{(i,j)}\). However, as we shall see shortly, it is still possible to extract all the necessarily forwards and discounting rates in the USD silo.

For USD domestic market, we can readily derive \(\{D^{(j)}(0, T_m), B^{(i,j)}(0, T_{m-1}, T_m)\}\) at every relevant point of time by applying the method explained in the previous sections. Suppose we have completed the curve construction for these USD instruments. Then, from Eq. (2.9.7) of MtMCCS, we can see that the only unknowns are the set of \(\{D^{(j)}(0, T_m)\}\), and hence we can bootstrap them using the market quotes of IRS \(\{S^{(j)}_M\}\) and MtMCCS.
\[ D^{(j)}(0, T_M) = \left\{ D^{(j)}(0, T_0) - \left( \frac{\Delta}{\delta} S^{(j)}_M + X_M \right) \sum_{m=1}^{M-1} \delta^{(j)}_m D^{(j)}(0, T_m) \right. \]
\[ \left. + \sum_{m=1}^{M} \delta^{(j)}_m D^{(j)}(0, T_m - 1) B^{(j)}(0; T_{m-1}, T_m) \overline{D}^{(j)}(0, T_{m-1}) / \left( 1 + \Delta^{(j)}_m S^{(j)}_M + \delta^{(j)}_m X_M \right) \right\} \]

which fixes \( D^{(j)}(0, T_M) \). Repeated use of the above calculation recursively determines all the relevant \( \{ \overline{D}^{(j)}(0, T_m) \} \).

Once we obtain the effective discounting factors of currency \((j)\), it is straightforward to obtain the associated LIBOR forwards. As before, let us suppose we have fixed \( \{ E^{T^{(j)}_m}[L^{(j)}(T_{m-1}, T_m)] \}_{m=1}^{M-1} \). Then, it is easy to see that the next forward is determined by

\[
E^{T^{(j)}_m}[L^{(j)}(T_{M-1}, T_M)] = \left\{ S^{(j)}_M \sum_{m=1}^{M} \Delta^{(j)}_m \overline{D}^{(j)}(0, T_m) \right. \\
\left. - \sum_{m=1}^{M-1} \delta^{(j)}_m D^{(j)}(0, T_m) E^{T^{(j)}_m}[L^{(j)}(T_{m-1}, T_m)] / \left( \delta^{(j)}_M \overline{D}^{(j)}(0, T_M) \right) \right\} . \tag{2.9.10}
\]

In the real market, we may not have all the consecutive quotes. In this case, we need to perform a joint calibration of IRS and MtMCCS assuming some appropriate spline functions for interpolation.

### 2.9.4 Heath-Jarrow-Morton framework in the USD silo

It is straightforward to assign HJM framework for the USD silo term structures. Firstly, let us define the instantaneous forward rate for the USD collateralized discounting rate:

\[
E^{Q^{(j)}}_t \left[ e^{-\int_t^T \overline{r}^{(j)}(s)ds} \right] = e^{-\int_t^T \overline{r}^{(j)}(s)ds} = E^{Q^{(j)}}_t \left[ e^{-\int_t^T \overline{r}^{(j)}(s)ds} \right] \]

or equivalently

\[
\overline{r}^{(j)}(t, T) = -\frac{\partial}{\partial T} \ln \overline{D}^{(j)}(t, T) . \tag{2.9.12}
\]

Note that, this rate can be negative when the CCS basis spread is significantly negative. By following the same arguments in Section 2.2.1, we can see its dynamics can be written in the following form in general:

\[
d\overline{r}^{(j)}(t, s) = \sigma^{(j)}(t, s) \cdot \left( \int_t^s \sigma^{(j)}(t, u)du \right) dt + \sigma^{(j)}(t, s) \cdot dW^{Q^{(j)}}_t , \tag{2.9.13}
\]

where \( W^{Q^{(j)}} \in \mathbb{R}^d \) is the \( d \)-dimensional \( Q^{(j)} \) Brownian motion, and \( \sigma^{(j)}(\cdot, s) \in \mathbb{R}^d \) is some appropriate \( d \)-dimensional adapted process.
Now, let us consider the forward LIBOR $L^{(j)}_{m}(t) := E^{(j)}_{t} \left[ L^{(j)}(T_{m-1}, T_{m}) \right]$. (2.9.14)

Since it should be a $\mathcal{F}^{(j)}_{m}$-martingale, it can be expressed as

$$dL^{(j)}_{m}(t) = \Sigma^{(j)}_{m}(t) \cdot \left( \int_{t}^{T_{m}} \sigma^{(j)}(t, u) du \right) dt + \Sigma^{(j)}_{m}(t) \cdot dW^{Q^{(j)}}_{t}.$$ (2.9.15)

using some appropriate $\Sigma^{(j)}_{m}(\cdot) \in \mathbb{R}^{d}$ adapted process. Note that the Brownian motion under the USD collateralized forward measure $\mathcal{F}^{(j)}_{m}$ is related to the standard money-market measure by

$$dW^{\mathcal{F}^{(j)}_{m}}_{t} = dW^{Q^{(j)}}_{t} + \left( \int_{t}^{T_{m}} \sigma^{(j)}(t, u) du \right) dt.$$ (2.9.16)

For the FX modeling, it is easy to check that the arbitrage free dynamics of the spot FX of currency $(j)$ relative to USD (or $i$) can be specified as

$$d f^{(j,i)}_{x}(t) / f^{(j,i)}_{x}(t) = \left( \tau^{(j)}(t) - c^{(i)}(t) \right) dt + \sigma^{(j,i)}_{X}(t) \cdot dW^{Q^{(j)}}_{t},$$ (2.9.17)

Similarly, the FX rate dynamics between the two emerging currencies $(j,k)$ in the USD silo is

$$d f^{(j,k)}_{x}(t) / f^{(j,k)}_{x}(t) = \left( \tau^{(j)}(t) - \tau^{(k)}(t) \right) dt + \sigma^{(j,k)}_{X}(t) \cdot dW^{Q^{(j)}}_{t}.$$ (2.9.18)

This is easy to check by noticing that the drift of spot FX is equivalent to the difference of risk-free interest rate of the two currencies in both cases. It is important to emphasize that there is no direct need to model risk-free interest rate also in this USD silo framework. All the necessary ingredients of the model can be observed in the market.

### 2.9.5 A possible adjustment for non-USD collateral

In SCSA, some of emerging currencies as well as multi-currency trades will be assigned to the USD silo and collateralized by the USD cash. However, if USD is the unique eligible collateral, it may induce liquidity squeeze and raise the USD funding cost under stressed market conditions. In order to mitigate the problem, it is being considered to accept also a different currency with an appropriate adjustment mechanism that guarantees the same economic effects as the standard case of USD collateralization.

If the option $h^{(j)}$ is collateralized by the USD cash, we know that its value is given by

$$h^{(j)}(t) = E^{Q^{(j)}}_{t} \left[ \exp \left( - \int_{t}^{T} \tau^{(j)}(s) ds \right) h^{(j)}(T) \right].$$ (2.9.19)

where $\tau^{(j)} = \tau^{(j)} - y^{(i)}$. Suppose now that the same option is collateralized by the payoff currency $(j)$ with some arbitrary collateral rate $\tilde{c}^{(j)}$. In this case, one obtains the option value as

$$\tilde{h}^{(j)}(t) = E^{Q^{(j)}}_{t} \left[ \exp \left( - \int_{t}^{T} \tilde{c}^{(j)}(s) ds \right) h^{(j)}(T) \right].$$ (2.9.20)
Thus, if the adjusted collateral rate satisfies
\[ \tilde{c}^{(j)}(t) = \tau^{(j)}(t) \] (2.9.21)
we can recover the same price as in the USD collateralization.

Thus, the only problem is to find a simple way to calculate such \( \tilde{c}^{(j)} \) from the available market data. If we assume the existence of short-term FX swaps, we can observe short-term FX forward points in the market. Consider, for example, a spot-start (spot+one business date)-maturing FX swap between the currency \((j)\) and USD. Since we know the USD overnight rate (Fed-fund rate) for the corresponding period, we can extract \( \tau^{(j)} \) for the same period from the market FX forward point. This is because the USD collateralized FX-forward is given by Eq. (2.8.8). The proper choice of FX swap depends on the details of settlement schedules of the relevant collateral account to be modeled. For consistency, these FX swaps should be continuously collateralized by USD, but in the market, very shortFX swaps may not have margin calls. However, since it involves the initial notional exchange, it seems as if one party obtains different currency by posting the equivalent amount of USD, which makes short FX swaps approximately USD collateralized.

Let us consider what should be done if another currency \(k\) in the USD silo is posted as collateral with the collateral rate \(\tilde{c}^{(k)}\) for the same option. Under this situation, the option value can be expressed as
\[ \tilde{h}^{(j)}(t) = E_t^{Q^{(j)}} \left[ \exp \left( - \int_t^T (\tau^{(j)}(s) - \tilde{y}^{(k)}(s)) \, ds \right) \tilde{h}^{(j)}(T) \right] \] (2.9.22)
where
\[ \tilde{y}^{(k)}(t) = r^{(k)}(t) - \tilde{c}^{(k)}(t) \] (2.9.23)
Hence, in order to recover the same price as the standard USD collateralization, the collateral rate for the currency \(k\) should satisfy \( \tilde{y}^{(k)} = y^{(i)} \), i.e.,
\[ \tilde{c}^{(k)}(t) = r^{(k)}(t) - y^{(i)} = \tau^{(k)} \] (2.9.24)
this is nothing but the effective discounting rate of currency \((k)\) under USD collateralization. Therefore, it can be extracted by using the market quotes USD and currency \(k\) FX swaps following the previous argument.

**Remark**
As we have seen, the existence of liquid short-term (such as tomorrow-next and spot-next) FX swap markets will be enough for the daily adjustment of collateral fee. However, in order to guarantee the transparency of general valuation of derivatives, we also need a long-term CCS to derive a forward expectation of the relevant rates. Replacing FX swaps by repo transactions, it is easy to understand that the method explained above can be extended to non-cash collaterals, too.

### 2.10 Conclusion

In this chapter, we have discussed the interest rate modeling framework under the full collateralization assuming no counterparty risk remains. We have explained a dynamic
term structure modeling in a multi-currency environment by using Heath-Jarrow-Morton
(HJM) framework. We have paid particular attention to the consistency with cross cur-
rency markets and provided the systematic procedures for the curve construction. The
importance of choice of collateral currency and the associated optionality in CSA were
emphasized with some numerical examples. We have also briefly mentioned about the
forthcoming SCSA and a possible pricing framework under the USD silo.
Chapter 3

Pricing under Asymmetric and/or Imperfect Collateralization

3.1 Introduction

In the last chapter, modeling of interest rate term structures under collateralization has been studied based on our series of works [10, 11, 12, 14], where cash collateral is assumed to be posted continuously and hence the remaining counterparty credit risk is negligibly small. It was found that there exists a direct link between the cost of collateral and CCS (cross currency swap) spreads. In fact, one cannot neglect the cost of collateral to make the whole system consistent with CCS markets, or equivalently with FX forwards. Making use of this relation, we have also observed the significance of a "cheapest-to-deliver" (CTD) option implicitly embedded in a collateral agreement. In this chapter, we further extend the previous framework so that we can handle asymmetric as well as imperfect collateralization in the unified credit-risk modeling framework.

Asymmetric collateralization arises when CSA (credit support annex) treats the two contracting parties asymmetrically, such as different collateral thresholds and unilateral collateralization. Even if the adopted CSA is symmetric, asymmetric collateralization may arise due to the different level of sophistication in collateral management of the two parties. For example, even if CSA allows the same set of eligible currencies for both parties, if one of them does not have appropriate system or easy access to the relevant foreign currencies, it cannot fully exercise the allowed optionality of choosing the cheapest currency to post. We give interesting numerical examples using OIS (overnight index swap) and CCS.

By introducing the collateral coverage ratio, our pricing framework allows to include under- as well as over-collateralization where there remains counterparty credit risk. In general, we see the pricing formula becomes non-linear FBSDEs since the amount of collateral and also the counter-party exposure depend on the value of portfolio at every time in the future, which itself is affected by the collateral cost and default payoff in turn. We have adopted Gateaux derivative to obtain the first order approximation, as in the work of Duffie & Huang (1996) [7], and found that it is possible to decompose the price of portfolio into the following parts:

1. Clean price representing the value under the perfect and symmetric collateralization.
(2) Bilateral CVA (or CVA/DVA).
(3) CCA (collateral cost adjustment) representing the change of collateral cost due to the
deviation from the perfect symmetric collateralization but independent from credit risk.

We will see that the clean price under the perfect and symmetric collateralization has the
desirable features as the market benchmark. It retains the additivity and hence one can
evaluate each trade (and cash flow) separately. In fact, we see that the clean price formula
is consistent with those obtained in Chapter 2, which is becoming the market standard
as the OIS discounting [21]. In other words, our pricing method allows one to decompose
the value of a generic contract into the market benchmark and the remaining corrections
which are credit risk and collateral cost.

The organization of Chapter 3 is as follows: In Sec. 3.2, we discuss the generic formu-
alation. The first order approximation using Gateaux derivative is explained in Sec. 3.3.
Secs. 3.4 and 3.5 are devoted to explain the case of perfect collateralization and Sec. 3.6
provides numerical examples to demonstrate the effects of asymmetric collateralization.
Sec. 3.7 discusses the implications for the behavior of financial firms induced by these
effects. Sec. 3.8 treats the imperfect collateralization and CVA, which shows the impor-
tance of the interplay between the collateral funding cost and other variables. We finally
conclude in Sec. 3.10. The Appendix contains technical details and proofs omitted in the
main text.

3.2 Generic Formulation

In this section, we consider the generic pricing formula. As an extension from the previous
works, we allow asymmetric and/or imperfect collateralization with bilateral default risk.
We basically follow the setup in Duffie & Huang (1996) [7] and extend it so that we can
deal with existence of collateral and its cost explicitly. The approximate pricing formulas
that allow simple analytic treatment are derived by Gateaux derivatives. See other related
developments in, for examples, [1, 4, 23, 2, 22] and references therein.

3.2.1 Fundamental Pricing Formula

Setup

We consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, Q)\), where \(\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}\) is sub-\(\sigma\)-
algebra of \(\mathcal{F}\) satisfying the usual conditions. Here, \(Q\) is the spot martingale measure, where
the money market account is being used as the numeraire. We consider two counterparties,
which are denoted by party 1 and party 2. We model the stochastic default time of party \(i\)
\((i \in \{1, 2\})\) as an \(\mathbb{F}\)-stopping time \(\tau^i \in [0, \infty]\), which are assumed to be totally inaccessible.
We introduce, for each \(i\), the default indicator function, \(H^i_t = 1_{\{\tau^i \leq t\}}\), a stochastic process
that is equal to one if party \(i\) has defaulted, and zero otherwise. The default time of
any financial contract between the two parties is defined as \(\tau = \tau^1 \wedge \tau^2\), the minimum
of \(\tau^1\) and \(\tau^2\). The corresponding default indicator function of the contract is denoted by
\(H_t = 1_{\{\tau \leq t\}}\). The Doob-Meyer theorem implies the existence of the unique decomposition
as \(H^i = A^i + M^i\), where \(A^i\) is a predictable and right-continuous (it is continuous indeed,
since we assume total inaccessibility of default time), increasing process with \(A^i_0 = 0\),
and $M^i$ is a $Q$-martingale. In the following, we also assume the absolute continuity of $A^i$ and the existence of progressively measurable non-negative process $h^i$, usually called the hazard rate of counterparty $i$, such that

$$A^i_t = \int_0^t h^i_s \mathbf{1}_{\{\tau^i > s\}} ds, \quad t \geq 0.$$  

(3.2.1)

For simplicity we also assume that there is no simultaneous default with positive probability and hence the hazard rate for $H_t$ is given by $h_t = h^1_t + h^2_t$ on the set of $\{\tau > t\}$.

**Collateralization**

We assume collateralization by cash which works in the following way: *if the party $i$ ($\in \{1, 2\}$) has negative mark-to-market, it has to post cash collateral $^1$ to the counterparty $j$ ($\neq i$), where the coverage ratio of the exposure is denoted by $\delta^i_t \in \mathbb{R}_+$. We assume the margin call and settlement occur instantly. Party $j$ is then a collateral receiver and has to pay collateral rate $c^i_t$ on the posted amount of collateral, which is $\delta^i_t \times \|\text{mark-to-market}\|$, to the party $i$. This is done continuously until the end of the contract. A common practice in the market is to set $c^i_t$ as the time-$t$ value of overnight (ON) rate of the collateral currency used by the party $i$.*

We emphasize that it is crucially important to distinguish the ON rate $c^i_t$ from the theoretical risk-free rate of the same currency $r^i_t$, where both of them are progressively measurable. The distinction is necessarily for the unified treatment of different collaterals and for the consistency with cross currency basis spreads, or equivalently FX forwards in the market (See, Sec. 3.5.4 and Chapter 2 for details.).

We consider the assumption of continuous collateralization is a reasonable proxy of the current market where daily (even intra-day) margin call is becoming popular. Although we assume continuous collateralization, we include the under- as well as over-collateralization in which we have $\delta^i_t < 1$ and $\delta^i_t > 1$, respectively. It may look slightly odd at first sight to include $\delta^i_t \neq 1$ cases under the continuous assumption, we think that allowing it makes the model more realistic considering the possible price dispute between the relevant parties, which is particularly the case for exotic derivatives. Because of the model uncertainty, the price reconciliation is usually done in ad-hoc way, say taking an average of each party’s quote. As a result, even after the each margin settlement, there always remains sizable discrepancy between the collateral value and the model implied fair value of the portfolio. Furthermore, in order to prepare for the rapid change of the collateral value, over-collateralization with sizable haircut is also frequently observed in the market.

Under these assumptions, the remaining credit exposure of the party $i$ to the party $j$ at time $t$ is given by

$$\max(1 - \delta^j_t, 0) \max(V^i_t, 0) + \max(\delta^j_t - 1, 0) \max(-V^i_t, 0),$$

where $V^i_t$ denotes the mark-to-market value of the contract from the view point of party $i$. The second term corresponds to the over-collateralization, where the party $i$ can only

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$^1$According to the ISDA survey [18], more than 80% of collateral being used is cash. If there is a liquid repo or security-lending market, we may also carry out similar formulation with proper adjustments of its funding cost.
recover the fraction of overly posted collateral when party \( j \) defaults. We denote the recovery rate of the party \( j \), when it defaults at time \( t \), by the progressively measurable process \( R^j_t \in [0,1] \). Thus, the recovery value that the party \( i \) receives can be written as

\[
R^i_t \left( \max(1 - \delta^j_t, 0) \max(V^i_t, 0) + \max(\delta^j_t - 1, 0) \max(-V^i_t, 0) \right), \quad (3.2.2)
\]

As for notations, we will use a bracket "( )" when we specify type of currency, such as \( r_t^{(i)} \) and \( c_t^{(i)} \), the risk-free and the collateral rates of currency \((i)\), in order to distinguish it from that of counterparty. We also denote a spot FX at time \( t \) by \( f^{(i,j)}_x(t) \) that is the price of a unit amount of currency \((j)\) in terms of currency \((i)\). We assume all the technical conditions for integrability are satisfied throughout this chapter.

**Pricing Formula**

We consider the ex-dividend price at time \( t \) of a generic financial contract made between the party 1 and 2, whose maturity is set as \( T (> t) \). We consider the valuation from the view point of party 1, and define the cumulative dividend \( D_t \) that is the total receipt from party 2 subtracted by the total payment from party 1. We denote the contract value as \( S_t \) and define \( S_t = 0 \) for \( \tau \leq t \). See Ref.[7] for the technical details about the regularity conditions which guarantee the existence and uniqueness of \( S_t \).

Under these assumptions and the setup give in Sec. 3.2.1, one obtains

\[
S_t = \beta_t E^Q \left[ \int_{[t,T]} \beta^{-1}_u 1_{\{\tau > u\}} \left\{ dD_u + (y_u^1 \delta^1_u 1_{\{S_u < 0\}} + y_u^2 \delta^2_u 1_{\{S_u \geq 0\}}) S_u du \right\} \right. \bigg| F_t \bigg] \quad (3.2.3)
\]

on the set of \( \{ \tau > t \} \). Here, \( y^i = r^i - c^i \) denotes a spread between the risk-free and collateral rates of the currency used by party \( i \), which represents the instantaneous return from the collateral being posted, i.e. it earns \( r^i \) but subtracted by \( c^i \) as the payment to the collateral payer. Here, we have used the risk-free rate as the effective investment return or borrowing cost of cash after adjusting all the market and credit risks. \( \beta_t = \exp \left( \int_0^t r_u du \right) \) is a money market account for the currency on which \( S_t \) is defined. \( Z^i \) is the recovery payment from the view point of the party 1 at the time of default of party \( i \in \{ 1, 2 \} \):

\[
Z^1(t, v) = \left( 1 - (1 - R^1_t)(1 - \delta^1_t)^+ \right) v_{1\{v < 0\}} + \left( 1 + (1 - R^1_t)(\delta^2_t - 1)^+ \right) v_{1\{v \geq 0\}} \quad (3.2.4)
\]

\[
Z^2(t, v) = \left( 1 - (1 - R^2_t)(1 - \delta^2_t)^+ \right) v_{1\{v \geq 0\}} + \left( 1 + (1 - R^2_t)(\delta^1_t - 1)^+ \right) v_{1\{v < 0\}} \quad (3.2.5)
\]

where \( X^+ \) denotes max(\(X, 0\)). Note that the above definition is consistent with the setup in Sec.3.2.1. The surviving party loses money if the received collateral from the defaulted party is not enough or if the posted collateral to the defaulted party exceeds the fair contract value.

Eq. (3.2.3) contains the indicator function \( H^i \) within the expectation and is not useful for valuation. Thus, as usually done in the credit modeling, we try to eliminate the indicators from the expectation. Even in the presence of the collateral, it is in fact possible to prove the following proposition in completely parallel fashion with the one given in [7]:

\[42\]
Proposition 1 Suppose a generic financial contract between the party 1 and 2, of which cumulative dividend at time $t$ is denoted by $D_t$ from the viewpoint of the party 1. Assume that the contract is continuously collateralized by cash where the coverage ratio of the party $i$ ($i \in \{1, 2\}$’s exposure is denoted by $\delta_i t \in \mathbb{R}^+$. The collateral receiver $j$ has to pay the collateral rate denoted by $c_i^t$ on the amount of collateral posted by party $i$, which is not necessarily equal to the risk-free rate of the same currency, $r_i^t$. The fractional recovery rate $R_i^t \in [0, 1]$ is assumed for the under- as well as over-collateralized exposure. For the both parties, totally inaccessible default is assumed, and the hazard rate process of party $i$ is denoted by $h_i^t$. We assume there is no simultaneous default of the party 1 and 2, almost surely.

Then, conditioned on no-default ($\tau > t$), the contract value $S_t$ given in Eq. (3.2.3) is represented by the pre-default value $V_t$ satisfying $S_t = V_t 1_{\{\tau > t\}}$

$$V_t = E^Q \left[ \int_{[t,T]} \exp \left( -\int_t^s (r_u - \mu(u,V_u)) du \right) dD_s \right] , \ t \leq T \quad (3.2.6)$$

where

$$\mu(t, v) = \left( y_1^t \delta_1^t - (1 - R_1^t)(1 - \delta_1^t) + h_1^t \right) 1_{\{v < 0\}} + \left( y_2^t \delta_2^t - (1 - R_2^t)(1 - \delta_2^t) + h_2^t \right) 1_{\{v \geq 0\}} \quad (3.2.7)$$

if the jump of $V$ at the time of default ($= \tau$) is zero almost surely.

See Appendix A.1 for the proof.

Naively speaking, by focusing on the pre-default world ($\tau > t$), we have replaced all the indicator functions in (3.2.3) with the corresponding intensities (or hazard rates), which then leads to the expression of $V$ in (3.2.6) by straightforward integration. However, precisely speaking, we have to check if the two quantities $S_t$ and $V_t 1_{\{\tau > t\}}$ actually follow the equivalent SDEs up to the default time. This point is actually confirmed in the proof. Note that the no-jump condition $\Delta V_\tau = 0$ is not crucial. Since we are only interested in pre-default value, we can replace the hazard rates as those conditioned on no-default, which then easily recovers $\Delta V_\tau = 0$. See the remark in Appendix A.1 for the details.

3.3 Decomposing the Pre-default Value

From Proposition 1, one sees the effective discounting rate becomes non-linear due to the correction term $\mu(t, v)$:

$$\mu(t, v) = \tilde{y}^1 t 1_{\{v < 0\}} + \tilde{y}^2 t 1_{\{v \geq 0\}} \quad (3.3.1)$$

where

$$\tilde{y}^i_t = \delta_i^t y_i^t - (1 - R_i^t)(1 - \delta_i^t) + h_i^t \quad (3.3.2)$$

for $i \in \{1, 2\}$ and $j \neq i$. As one can see, due to the presence of the dependence on $v$ through the indicator, the pricing formula of Eq. (3.2.6) together with other market variables form a system of non-linear FBSDE. Since it is impossible to solve in general, we need to consider some approximation procedures.
3.3.1 Perfect and Symmetric Collateralization

Since non-linearity appears only through the indicators, we can recover the linearity if
\[ \tilde{y}_s^1 = \tilde{y}_s^2 \quad \text{Lebesgue-a.e.} \quad s \in [t, T] \quad \text{a.s.} \quad (3.3.3) \]

This situation naturally arises when the collateralization is perfect and symmetric, or \((\delta^1 = \delta^2 = 1)\) and \((y^1 = y^2 = \bar{y})\). In fact, in this case, the pricing formula (3.2.6) becomes
\[
V_t = E^Q \left. \exp \left( - \int_t^T (r_u - \bar{y}_u) \, du \right) \, dD_s \right| \mathcal{F}_t . \quad (3.3.4)
\]

Note that we have recovered the additivity here: The portfolio value can be obtained by adding that of each trade, which is calculable separately. This is the most important feature as the market benchmark since otherwise a trade price depends on the whole portfolio to the specific counterparty and there would be no price transparency. As we will see shortly, this actually corresponds to the collateral rate discounting [10, 25], which is becoming the market standard [21].

3.3.2 Generic Situations

Now, let us consider more generic situations where the collateralization is imperfect \((\delta^i \neq 1)\) and/or asymmetric \((y^i \neq y^2)\). In order to evaluate Eq. (3.2.6) approximately, we consider expanding \(V_t\) around the previously obtained \(V_t\) in the first order of Taylor expansion. It means that we try to express the contract value by the market benchmark and the remaining corrections.

Firstly, let us express the correction to the discounting rate \(\mu(t, v)\) around the collateral cost at the benchmark point \(\bar{y}\):
\[
\mu(t, v) = \bar{y}_t + \Delta \tilde{y}_t^1 1_{\{v < 0\}} + \Delta \tilde{y}_t^2 1_{\{v \geq 0\}} \quad (3.3.5)
\]

where
\[
\Delta \tilde{y}_t^i = \tilde{y}_t^i - \bar{y}_t^i = \left\{ (\delta^i \bar{y}_t^i - \bar{y}_t^i) - (1 - R_t^i)(1 - \delta^i)^+ h_t^i + (1 - R_t^j)(\delta^i - 1)^+ h_t^j \right\} \quad (3.3.6)
\]

for \(i \in \{1, 2\}\) and \(j \neq i\). The first order effect of \(\Delta \tilde{y}\) is given by Gateaux derivative \(\nabla V\), which is a sort of gradient by considering \(V\) as a functional of \(\tilde{y}\):
\[
\limsup_{\epsilon \downarrow 0} \sup_t \left| \nabla V(\bar{y}, \tilde{y}) - \frac{V_t(\bar{y} + \epsilon (\Delta \tilde{y}^1 1_{\{v < 0\}} + \Delta \tilde{y}^2 1_{\{v \geq 0\}})) - V_t(\bar{y})}{\epsilon} \right| . \quad (3.3.7)
\]

Notice that \(V_t(\bar{y})\) is actually \(\bar{V}_t\) in Eq. (3.3.4). Following the method explained in Duffie & Skiadas (1994) [8] and Duffie & Huang (1996) [7], we can derive
\[
\nabla V(\bar{y}, \tilde{y}) = E^Q \left[ \int_t^T e^{-f_t^i(r_u - \bar{y}_u) \, du} V_s \left( \Delta \tilde{y}^1_s 1_{\{v < 0\}} + \Delta \tilde{y}^2_s 1_{\{v \geq 0\}} \right) ds \right| \mathcal{F}_t . \quad (3.3.8)
\]
Substituting the contents of $\Delta \tilde{y}$ into (3.3.8), the above Gateaux derivative can be further decomposed into two parts, one of which is the collateral cost adjustment independent from the credit risk

$$CCA_t = E^Q \left[ \int_t^T e^{-\int_t^s (r_u - \tilde{y}_u) du} \left( - (\delta^1_s y^1_s - \tilde{y}_s) [-V_s]^+ + (\delta^2_s y^2_s - \tilde{y}_s) [V_s]^+ \right) ds \right] \mathcal{F}_t \right]$$

and the other is the well-known bilateral CVA terms

$$CVA_t = E^Q \left[ \int_t^T e^{-\int_t^s (r_u - \tilde{y}_u) du} h^1_s (1 - R^1_s) \left\{ (1 - \delta^1_s)^+ [-V_s]^+ + (\delta^2_s - 1)^+ [V_s]^+ \right\} ds \right] \mathcal{F}_t \right]$$

Using these terms, the original pre-default value of Eq. (3.2.6) can be approximated in the first order of $\Delta \tilde{y}$ as

$$V_t = V_t^0 + CCA_t + CVA_t + o(\Delta \tilde{y}^1, \Delta \tilde{y}^2) \, . \quad (3.3.10)$$

In the following sections, we will study each term more closely. We basically treat the perfect collateralization ($\delta^1 = \delta^2 = 1$) until Sec. 3.8, where we will study CVA.

### 3.4 Perfect Collateralization

In the following section, we first deal with the perfectly collateralized situation where $CVA = 0$. We check the several important examples necessary to price some fundamental instruments used in numeral examples in the next section.

#### 3.4.1 Symmetric Collateralization

Let us first consider the market benchmark, or the perfect and symmetric collateralization. In this case we have $\delta^1 = \delta^2 = 1$ and $y^1 = y^2 = \tilde{y}$, and hence $CCA = CVA = 0$. One can easily confirm that all the following results are consistent with those given in Chapter 2.

**Case 1:** Situation where both parties use the same collateral currency "(i)", which is the same as the deal currency. Since we have $\tilde{y} = y^{(i)} = r^{(i)} - \ell^{(i)}$, in this case, the pre-default value of the contract in terms of currency (i) is given by

$$V_t = E^{Q^{(i)}} \left[ \int_{[t,T]} \exp \left( - \int_t^s c_u^{(i)} du \right) dD_s \right] \mathcal{F}_t \right] , \quad (3.4.1)$$

where $Q^{(i)}$ is the spot-martingale measure of currency (i). In this case, we can use the collateral rate to discount the future cash flows as if it is the usual risk-free rate. Since the collateral rate of cash is the corresponding ON rate, this formula corresponds to the OIS discounting method, which is becoming the new standard in the market. \(^2\)

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\(^2\)See the recent survey done by KPMG [21].
Case 2: Situation where both parties use the same collateral currency "(k)", which is different from the deal currency "(i)". In this case, \( \bar{y} = y^{(k)} \) and the pre-default value of the contract in terms of currency (i) is given by

\[
V_t = E^{Q^{(i)}} \left[ \int_{\mathcal{F}_t}^{T} \exp \left( -\int_t^s \left( c^{(i)}_u + y^{(i,k)}_u \right) du \right) dD_s \bigg| \mathcal{F}_t \right] , \tag{3.4.2}
\]

where we have defined the funding spread between the currencies (i) and (k):

\[
y^{(i,k)}_u = y^{(i)}_u - y^{(k)}_u \tag{3.4.3}
\]

\[
y^{(i,k)}_u = \left( r^{(i)}_u - c^{(i)}_u \right) - \left( r^{(k)}_u - c^{(k)}_u \right) \tag{3.4.4}
\]

This formula is particularly important for non-G5 currencies and multi-currency trades where it is quite common to use USD as the collateral currency (k). The funding spread reflects the funding cost of USD relative to the deal currency (i), which is reflected in the corresponding cross currency basis swap. This point will be explained in Sec. 3.5.4 in details.

Case 3: Situation where the deal currency is (i) and the both parties have a common set of eligible collateral currencies denoted by \( \mathcal{C} \). Note that only the collateral payer at each time has the right to choose the collateral to post. In this case, \( \bar{y} = \min_{k \in \mathcal{C}} y^{(k)} \) and then we have

\[
V_t = E^{Q^{(i)}} \left[ \int_{\mathcal{F}_t}^{T} \exp \left( -\int_t^s \left( c^{(i)}_u + \max_{k \in \mathcal{C}} y^{(i,k)}_u \right) du \right) dD_s \bigg| \mathcal{F}_t \right] \tag{3.4.5}
\]

as the pre-default value of the contract in terms of currency (i).

Notice that the collateral payer will choose the currency (k) that minimizes the cost of collateral \( \min_{k \in \mathcal{C}} y^{(k)} \), which is equivalent to maximizing the effective discounting rate so that the payer achieves the smallest mark-to-market loss. The optionality is crucially depends on the volatility of CCS and can be numerically quite significant. See Ref. [12] for details.

3.4.2 Asymmetric Collateralization

We now consider the situation where CVA = 0 but there remains non-zero CCA due to the asymmetric collateralization \( y^1 \neq y^2 \).

Suppose the situation where the trade is perfectly collateralized (\( \delta^1 = \delta^2 = 1 \)) and the party 1 can choose the optimal currency from the eligible set denoted by \( \mathcal{C} \) or (\( y^1 = \min_{k \in \mathcal{C}} y^{(k)} \)), whereas the party 2 can only use the single currency (j) as collateral (\( y^2 = y^{(j)} \)). Assume the deal currency is (i). If we choose the center of expansion as \( \bar{y} = y^{(j)} \) (\( = y^2 \)) and then we have

\[
CCA_t = E^{Q^{(i)}} \left[ \int_t^T \exp \left( -\int_t^s \left( c^{(i)}_u + y^{(i,j)}_u \right) du \right) \left[ -V_s + \max_{k \in \mathcal{C}} y^{(j,k)}_s \right] \bigg| \mathcal{F}_t \right] \tag{3.4.6}
\]
where
\[ V_t = E^{Q(i)} \left[ \int_{[t,T]} \exp \left( -\int_t^s \left( c_{si}^{(i)} + y_{iu}^{(i,j)} \right) du \right) dD_s \mid \mathcal{F}_t \right] . \] (3.4.7)

Since only the party 1 has the cheapest-to-deliver optionality, the CCA adds the positive value to the contract \(^3\). This value should be reflected in the contract price otherwise the party 2 will suffer a loss by giving a free option to the party 1.

Although we have assumed the asymmetric collateral agreement in the above example, similar situation can naturally arise even if the relevant CSA itself is symmetric. For example, even if the party 2 has the same eligible collateral set, if it lacks the easy access to the currencies involved, then it ends up with the same situation. One can see that it is very dangerous to make a flexible collateral agreement if there is no ability to fully exercise its embedded optionality, especially when the counterparty is more sophisticated in collateral management. We will give interesting numerical studies for the above example in Sec. 3.6.

Another important example of asymmetric collateralization is the one-way CSA where collateralization is performed only unilaterally. This is actually common when sovereigns or central banks are involved as counterparties. The detailed explanation of this situation is given in Sec. 3.8 with default risk taken into account.

3.5 Some Fundamental Instruments

In order to study the quantitative effects of collateralization, we firstly need to understand the clean price, i.e., \( \overline{V} \). The details of generic term structure modeling under perfect collateralization are available in Refs [10, 11, 12]. In this section, we just summarize some of the fundamental instruments required to understand the following numerical examples which demonstrates the impact of asymmetric collateralization.

3.5.1 Collateralized Zero Coupon Bond

We define the collateralized zero coupon bond of currency \((i)\) as
\[ D^{(i)}(t, T) = E^{Q(i)} \left[ e^{-\int_t^T c_{si}^{(i)} ds} \mid \mathcal{F}_t \right] . \] (3.5.1)

We call it "Bond" by analogy with conventional interest rate models but it simply represents the present value of the perfectly collateralized contract that has the unit payment of cash in the future time \(T\) (See, Eq. (3.4.1)). As you can see in [12] for example, this plays the role as the discounting factor for the collateralized cash flow. This fact can be easily understood by noticing that there is additivity in price for the perfect and symmetric collateralization \(^4\).

In the same way, for the case where the deal and collateral currencies are different, \((i)\) and \((j)\) respectively, we define the foreign collateralized zero coupon bond \(D^{(i,j)}\) by
\[ D^{(i,j)}(t, T) = E^{Q(i)} \left[ e^{-\int_t^T c_{si}^{(i)} ds} \left( e^{-\int_t^T y_{iuj}^{(i,j)} ds} \right) \mid \mathcal{F}_t \right] . \] (3.5.2)

\(^3\)Remember that we are calculating the contract value from the view point of the party 1.

\(^4\)There is no point to issue collateralized zero coupon bond to raise cash from the market since the issuer has to post the same amount of cash as collateral to the buyer.
In particular, if \( c^{(i)} \) and \( y^{(i,j)} \) are independent, we have

\[
D^{(i,j)}(t, T) = D^{(i)}(t, T)e^{-\int_t^T y^{(i,j)}(t,s)ds},
\]

where

\[
y^{(i,j)}(t, s) = -\frac{\partial}{\partial s} \ln E^{Q(i)} \left[ e^{-\int_t^T y^{(i,j)}(t,s)du} | \mathcal{F}_t \right]
\]

denotes the forward \( y^{(i,j)} \) spread. As before one needs to understand this instrument as the present value of unit payment collateralized by a foreign currency.

3.5.2 Collateralized FX Forward

Because of the existence of collateral, FX forward transaction now becomes non-trivial. The precise understanding of the collateralized FX forward is crucial to deal with generic collateralized products. The definition of currency-(\( k \)) collateralized FX forward contract for the currency pair \((i, j)\) is as follows:\(^5\)

- **At the time of trade inception** \( t \), both parties agree to exchange \( K \) unit of currency \((i)\) with the one unit of currency \((j)\) at the maturity \( T \). Throughout the contract period, the continuous collateralization by currency \((k)\) is performed, i.e. the party who has negative mark-to-market needs to post the equivalent amount of cash in currency \((k)\) to the counterparty as collateral, and this adjustment is made continuously. The FX forward rate \( f^{(i,j)}_x(t, T; k) \) is defined as the value of \( K \) that makes the value of the above contract zero at the time of its inception.

By using the results of Sec. 3.4.1, \( K \) needs to satisfy the following relation:

\[
KE^{Q(i)} \left[ e^{-\int_t^T (c^{(i)}_s + y^{(i,k)}_s)ds} | \mathcal{F}_t \right] - f^{(i,j)}(t)E^{Q(i)} \left[ e^{-\int_t^T (c^{(j)}_s + y^{(j,k)}_s)ds} | \mathcal{F}_t \right] = 0 \quad (3.5.5)
\]

and hence the FX forward is given by

\[
f^{(i,j)}(t, T; k) = f^{(i,j)}(t) \frac{E^{Q(i)} \left[ e^{-\int_t^T (c^{(j)}_s + y^{(j,k)}_s)ds} | \mathcal{F}_t \right]}{E^{Q(i)} \left[ e^{-\int_t^T (c^{(i)}_s + y^{(i,k)}_s)ds} | \mathcal{F}_t \right]} \quad (3.5.6)
\]

\[
f^{(i,j)}(t, T; k) = f^{(i,j)}(t) \frac{D^{(j,k)}(t, T)}{D^{(i,k)}(t, T)}, \quad (3.5.7)
\]

which becomes a martingale when \( D^{(i,k)}(\cdot, T) \) is used as the numeraire. In particular, we have

\[
E^{Q(i)} \left[ e^{-\int_t^T (c^{(i)}_s + y^{(i,k)}_s)ds} f^{(i,j)}_x(T) | \mathcal{F}_t \right] = D^{(i,k)}(t, T)E^{T^{(i,k)}} \left[ f^{(i,j)}_x(T, T; k) | \mathcal{F}_t \right] = D^{(i,k)}(t, T)f^{(i,j)}_x(t, T; k). \quad (3.5.8)
\]

\(^5\)In the market, USD is popular as the collateral for multi-currency trades.
Here, we have defined the \((k)\)-collateralized \((i)\) forward measure \(T^{(i,k)}\), where \(D^{(i,k)}(\cdot,T)\) is used as the numeraire. \(E_T^{(i,k)}[\cdot]\) denotes expectation under this measure.

**Remark:** In the case of FX futures, the trade is continuously settled and hence the gain or loss is immediately realized. Since the realized gain or loss is not considered as collateral, there is no exchange of collateral rate \(c\) between the financial firm and security exchange. Therefore, unless we set \(c = 0\), the collateralized FX forward value is different from that of futures.

### 3.5.3 Overnight Index Swap

The overnight index swap (OIS) is a fixed-vs-floating swap which is the same as the usual IRS except that the floating leg pays periodically, say quarterly, compounded ON rate instead of Libors. Let us consider \(T_0\)-start \(T_N\)-maturing OIS of currency \((j)\) with fixed rate \(S_N\), where \(T_0 \geq t\). If the party 1 takes a receiver position, we have

\[
dD_s = \sum_{n=1}^{N} \delta_{T_n}(s) \left( \Delta_n S_N + 1 - \exp \left( \int_{T_{n-1}}^{T_n} c_u^{(j)} du \right) \right)
\]

where \(\Delta\) is day-count fraction of the fixed leg, and \(\delta_T(\cdot)\) denotes Dirac delta function at \(T\). In particular, if OIS is collateralized by its domestic currency \((j)\), its value \(V_t\) is given by

\[
V_t = \sum_{n=1}^{N} \Delta_n D^{(j)}(t, T_n) S_N - \left( D^{(j)}(t, T_0) - D^{(j)}(t, T_N) \right)
\]

and hence the par rate is expressed as

\[
S_N = \frac{D^{(j)}(t, T_0) - D^{(j)}(t, T_N)}{\sum_{n=1}^{N} \Delta_n D^{(j)}(t, T_n)}.
\]

### 3.5.4 Cross Currency Swap

Cross currency swap (CCS) is one of the most fundamental products in FX market. Especially, for maturities longer than a few years, CCS is much more liquid than FX forward contract and is the dominant funding source of foreign currencies. The current market is dominated by USD crosses where 3m USD Libor flat is exchanged by 3m Libor of a different currency with additional (negative in many cases) basis spread. The most popular type of CCS is called MtMCCS (Mark-to-Market CCS) in which the notional of USD leg is refreshed at the every Libor fixing time, while the notional of the other leg is kept constant throughout the contract. For model calibration, MtMCCS should be used as we have done in Ref. [12] considering its liquidity. However, in the following, we study another type of CCS, which is actually tradable in the market, to make the link between \(y\) and CCS much clearer.

We study the Mark-to-Market cross currency overnight index swap (MtMCCOIS), which is exactly the same as the usual MtMCCS except that it pays a compounded ON rate, instead of the Libor, of each currency periodically. Let us consider \((i, j)\)-MtMCCOIS where currency \((i)\) intended to be USD and needs notional refreshments, and currency \((j)\)
is the one in which the basis spread is to be paid. Let us suppose the party 1 is the spread receiver and consider \( T_0 \)-start \( T_N \)-maturing \((i, j)\)-MtMCCOIS. For the \((j)\)-leg, we have

\[
dD_s^{(j)} = -\delta_{T_0}(s) + \delta_{T_N}(s) + \sum_{n=1}^{N} \delta_{T_n}(s) \left[ e^{\int_{T_n-1}^{T_n} \tau^{(j)} du} - 1 \right] + \delta_n B_N, \tag{3.5.12}
\]

where \( B_N \) is the basis spread of the contract. For \((i)\)-leg, in terms of currency \((i)\), we have

\[
dD_s^{(i)} = \sum_{n=1}^{N} \left[ \delta_{T_{n-1}}(s) f_x^{(i,j)}(T_{n-1}) - \delta_{T_n}(s) f_x^{(i,j)}(T_{n-1}) e^{\int_{T_{n-1}}^{T_n} \tau^{(i)} du} \right]. \tag{3.5.13}
\]

In total, in terms of currency \((j)\), we have

\[
dD_s = dD_s^{(j)} + f_x^{(j,i)}(s) dD_s^{(i)} \tag{3.5.14}
\]

\[
dD_s = dD_s^{(j)} + \sum_{n=1}^{N} \left[ \delta_{T_{n-1}}(s) - \delta_{T_n}(s) \frac{f_x^{(j,i)}(T_{n-1})}{f_x^{(i,j)}(T_{n-1})} \right] e^{\int_{T_{n-1}}^{T_n} \tau^{(j)} du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(i,j)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} \tau^{(i)} du}. \tag{3.5.15}
\]

If the collateralization is done by currency \((k)\), then the value for the party 1 is given by

\[
V_t = \sum_{n=1}^{N} \mathbb{E} \left[ Q^{(j)} \left[ e^{-\int_{t}^{T_n} \tau^{(j,i)}(t,u) du} \left\{ e^{\int_{T_{n-1}}^{T_n} \tau^{(i)} du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(i,j)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} \tau^{(i)} du} \right\} \right] \bigg| \mathcal{F}_t \right], \tag{3.5.17}
\]

where \( T_0 \geq t \). In particular, if the swap is collateralized by currency \((i)\) (or USD) that is popular in the market, we obtain

\[
V_t = \sum_{n=1}^{N} \delta_n B_N D^{(j)}(t, T_n) e^{-\int_{t}^{T_n} \tau^{(j,i)}(t,u) du} - \sum_{n=1}^{N} D^{(j)}(t, T_{n-1}) e^{-\int_{t}^{T_{n-1}} \tau^{(j,i)}(t,u) du} \left( 1 - e^{-\int_{T_{n-1}}^{T_n} \tau^{(j,i)}(t,u) du} \right)
\]

\[
= \sum_{n=1}^{N} \left[ \delta_n B_N D^{(j,i)}(t, T_n) - D^{(j,i)}(t, T_{n-1}) \left( 1 - e^{-\int_{T_{n-1}}^{T_n} \tau^{(j,i)}(t,u) du} \right) \right]. \tag{3.5.18}
\]

Here, we have assumed the independence of \( \tau^{(j)} \) and \( \tau^{(j,i)} \). In fact, the assumption seems reasonable according to the recent historical data studied in Ref. [12]. In this case, we obtain the par MtMCCOIS basis spread as

\[
B_N = \frac{\sum_{n=1}^{N} D^{(j,i)}(t, T_{n-1}) \left( 1 - e^{-\int_{T_{n-1}}^{T_n} \tau^{(j,i)}(t,u) du} \right)}{\sum_{n=1}^{N} \delta_n D^{(j,i)}(t, T_n)}. \tag{3.5.19}
\]
Thus, it is easy to see that

\[ B_N \sim \frac{1}{T_N - T_0} \int_{T_0}^{T_N} y^{(j,i)}(t, u) du, \]  

which gives us the relation between the currency funding spread \( y^{(j,i)} \) and the observed cross currency basis. Therefore, the cost of collateral \( y \) is directly linked to the dynamics of CCS markets. It is interesting to understand the origin of the funding spread \( y^{(i,k)} \) in the pricing formula (3.4.2) based on the above discussion of CCS. Interested readers can read Appendix A.2 for the details.

### 3.6 Numerical Studies for Asymmetric Collateralization

In this section, we study the effects of perfect but asymmetric collateralization and hence \((\text{CVA} = 0, \text{CCA} \neq 0)\), using the two fundamental products, MtMCCOIS and OIS. The results will clearly tell us that the sophistication of collateral management does matter in the real business. For both cases, we use the following dynamics in Monte Carlo simulation:

\[
dc_t^{(j)} = \left( \theta^{(j)}(t) - \kappa^{(j)} c_t^{(j)} \right) dt + \sigma_c^{(j)} dW_t^{1} \\
dc_t^{(i)} = \left( \theta^{(i)}(t) - \rho_{2,i} \sigma^{(j)} c_t^{(j)} - \kappa^{(i)} c_t^{(i)} \right) dt + \sigma^{(i)} dW_t^{2} \\
dy_t^{(j,i)} = \left( \theta^{(j,i)}(t) - \kappa^{(j,i)} y_t^{(j,i)} \right) dt + \sigma_y^{(j,i)} dW_t^{3} \\
d \ln f_{x}^{(j,i)}(t) = \left( c_t^{(j)} - c_t^{(i)} + y_t^{(j,i)} - \frac{1}{2} (\sigma_x^{(j,i)})^2 \right) dt + \sigma_x^{(j,i)} dW_t^{4} \]  

where \( \{W^i, i = 1 \ldots 4\} \) are Brownian motions under the spot martingale measure of currency \( (j) \). Every \( \theta(t) \) is a deterministic function of time, and is adjusted in such a way that we can recover the initial term structures of the relevant variable. The procedures for the curve construction are given in Appendix A.3. We assume every \( \kappa \) and \( \sigma \) are constants. We allow general correlation structure \((d[W^i, W^j]_t = \rho_{i,j} dt)\) except that \( \rho_{3,j} = 0 \) for all \( j \neq 3 \).

The above dynamics is chosen just for simplicity and demonstrative purpose, and generic HJM framework can also be applied to the evaluation of Gateaux derivative. For details of generic dynamics in HJM framework, see Chapter 2. In the following, we use the term structure for the \((i,j)\) pair taken from the typical data of (USD, JPY) in early 2010 for presentation. In Appendix A.8, we have provided the term structures and other parameters used in simulation.

#### 3.6.1 Asymmetric Collateralization for MtMCCOIS

We now implement Gateaux derivative using Monte Carlo simulation based on the model we have just explained. To see the accuracy of Gateaux derivative, we have compared it with a numerical result directly obtained from PDE using a simplified setup in Appendix A.7.

Firstly, we consider MtMCCOIS explained in Sec. 3.5.4. We consider a spot-start, \( T_N\)-maturing \((i,j)\)-MtMCCOIS, where the leg of currency \((i)\) (intended to be USD) needs
Figure 3.1: Price difference from symmetric limit for 10y MtMCCOIS

Figure 3.2: Price difference from symmetric limit for 20y MtMCCOIS
notional refreshments. Let us assume perfect but asymmetric collateralization as follows:
(1) Party 1 can use either the currency \((i)\) or \((j)\) as collateral.
(2) Party 2 can only use the currency \((i)\) as collateral.
For the derivation of the present value, see Appendix A.4.1.

In Figs. 3.1 and 3.2, we have shown the numerical result of CCA which is the price difference from the symmetric limit, for 10y and 20y MtMCCOIS, respectively. The spread \(B\) was chosen in such way that the value in symmetric limit, \(V_0\), becomes zero. In both cases, the horizontal axis is the annualized volatility of \(y^{(j,i)}\), and the vertical one is the price difference from CCA in terms of bps of notional of currency \((j)\). When the party 1 is the spread payer (receiver), we have used the left (right) axis. The results are rather insensitive to the FX volatility due to the notional refreshments of currency-\((i)\) leg. From the historical analysis performed in Ref. [12], we know that annualized volatility of \(y^{(j,i)}\) tends to be 50bps or so in a calm market, but it can be \((100 \sim 200)\)bps or more in a volatile market for major currency pairs, such as (EUR,USD) and (USD, JPY). Therefore, the impact of asymmetric collateralization in this example can be practically very significant when party 1 is the spread payer. When the party 1 is the spread receiver, one sees that the impact of asymmetry is very small, only a few bps of notional. This can be easily understood in the following way: When the party 1 has a negative mark-to-market and hence is the collateral payer who has the option to change the collateral currency, \(y^{(j,i)}\) tends to be large and hence the optimal currency remains the same currency \((i)\).

Finally, let us briefly mention about the standard MtMCCS with Libor payments. As discussed in Ref. [12], the contribution from Libor-OIS spread to CCS is not significant relative to that of \(y^{(j,i)}\). Therefore, the numerical significance of asymmetric collateralization is expected to be quite similar in the standard case, too.

### 3.6.2 Asymmetric Collateralization for OIS

Now we study the impact of asymmetric collateralization on OIS. We consider OIS of currency \((j)\), and assume the following asymmetry in collateralization:
(1) Party 1 can use either the currency \((i)\) or \((j)\) as collateral.
(2) Party 2 can only use the currency \((j)\) (domestic currency) as collateral.
For the derivation of the present value, see Appendix A.4.2.

In Figs. 3.3, 3.4, and 3.5, we have shown the numerical results of CCA for 10y and 20y OIS from the view point of party 1. In the first two figures, we have fixed \(\sigma^c_{(j)} = 1\%\) and changed \(\sigma^y_{(j,i)}\) to see the sensitivity against CCS. In the last figure, we have fixed the \(y^{(j,i)}\) volatility as \(\sigma^y_{(j,i)} = 0.75\%\) and changed the volatility of collateral rate \(c_{(j)}\). Since the term structure of OIS used in simulation is upward sloping, the mark-to-market value of the fixed receiver tends to be negative in the long end of the contract. This makes the cheapest-to-deliver optionality bigger for the receiver, and hence it has bigger CCA contribution than the case of payer.

### 3.7 General Implications of Asymmetric Collateralization

From the results of section 3.6, we have seen the practical significance of asymmetric collateralization. It is now clear that sophisticated financial firms may obtain significant
Figure 3.3: Price difference from symmetric limit for 10y OIS

Figure 3.4: Price difference from symmetric limit for 20y OIS

Figure 3.5: Price difference from symmetric limit for 20y OIS for the change of $\sigma_c^{(j)}$
funding benefit from the less-sophisticated counterparties.

Before going to discuss the imperfect collateralization and associated CVA, let us explain two generic implications of asymmetric collateralization, one for netting and the other for resolution of information, which is closely related to the observation just explained. Although derivation itself can be done in exactly the same way as Ref. [7] after the reinterpretation of several variables, we get new insights for collateralization that may be important for the appropriate design and regulations for the financial market.

3.7.1 An implication for Netting

Proposition 2  
Assume perfect collateralization. Suppose that, for each party i, \( y_i \) is bounded and does not depend on the contract value directly. Let \( V^a, V^b, \) and \( V^{ab} \) be, respectively, the value processes (from the view point of party 1) of contracts with cumulative dividend processes \( D^a, D^b, \) and \( D^a + D^b \) (i.e., netted portfolio). If \( y_1 \geq y_2 \), then \( V^{ab} \geq V^a + V^b \), and if \( y_1 \leq y_2 \), then \( V^{ab} \leq V^a + V^b \).

Proof is available in Appendix A.5. The interpretation of the proposition is very clear: The party who has the higher funding cost \( y \) due to asymmetric CSA or lack of sophistication in collateral management is expected to prefer to have netting agreements to decrease funding cost. On the other hand, an advanced financial firm who has capability to carry out optimal collateral strategy to achieve the lowest possible value of \( y \) prefers to avoid netting to exploit funding benefit if the other conditions are kept unchanged. For example, an advanced firm may prefer to enter an opposite trade with a different counterparty rather than to unwind the original trade. For standardized products traded through CCPs, such a firm may prefer to use several clearing houses cleverly to avoid netting.

The above finding seems slightly worrisome for the healthy development of CCPs. Advanced financial firms that have sophisticated financial technology and operational system are usually primary members of CCPs, and some of them are trying to set up their own clearing service facility. If those firms try to exploit funding benefit, they avoid concentration of their contracts to major CCPs and may create very disperse interconnected trade networks and may reduce overall netting opportunity in the market. Although remaining credit exposure is very small as long as collateral is successfully being managed, the dispersed use of CCPs may worsen the systemic risk once it fails. In the work of Duffie & Huang [7], the corresponding proposition is derived in the context of bilateral CVA. We emphasize that one important practical difference is the strength of incentives provided to financial firms. Although it is somewhat obscure how to realize profit/loss reflected in CVA, it is rather straightforward in the case of collateralization by making use of CCS market as we have explained in Sec. 3.5.4.

3.7.2 An implication for Resolution of Information

We once again follow the setup given in Ref [7]. We assume the existence of two markets: One is market \( F \), which has filtration \( \mathbb{F} \), that is the one we have been studying. The other

\[ \delta y_i > (1 - R_i(1 - \delta_i))(\delta_i - 1)^+ h_i - (1 - R_i(\delta_i - 1))^+ h_i. \]
one is market $G$ with filtration $G = \{G_t : t \in [0, T]\}$. The market $G$ is identical to the
market $F$ except that it has earlier resolution of uncertainty, or in other words, $\mathcal{F}_t \subseteq G_t$
for all $t \in [0, T]$ while $\mathcal{F}_0 = G_0$. The spot martingale measure $Q$ is assumed to apply to
the both markets.

**Proposition 3** 7 Assume perfect collateralization. Suppose that, for each party $i$, $y^i$ is
bounded and does not depend on the contract value directly. Suppose that $r$, $y^1$ and $y^2$ are
adapted to both the filtrations $\mathcal{F}$ and $G$. The contract has cumulative dividend process $D$,
which is a semimartingale of integrable variation with respect to filtrations $\mathcal{F}$ and $G$. Let
$V^F$ and $V^G$ denote, respectively, the values of the contract in markets $F$ and $G$ from the
view point of party 1. If $y^1 \geq y^2$, then $V^F_0 \geq V^G_0$, and if $y^1 \leq y^2$, then $V^F_0 \geq V^G_0$.

Proof is available in Appendix A.6. The proposition may imply that the party who has
the higher effective funding cost $y$ either from the lack of sophisticated collateral manage-
tment technique or from asymmetric CSA would like to delay the information resolution
to avoid timely margin call from the counterparty. The opposite is true for advanced
financial firms which are likely to have advantageous CSA and/or sophisticated system.
The incentives to obtain funding benefit will urge these firms to provide mark-to-market
information of contracts to counterparties in timely manner, and seek early resolution of
valuation dispute to achieve funding benefit. Considering the privileged status of these
firms, the latter effects will probably be dominant in the market.

## 3.8 Imperfect Collateralization and CVA

As we have explained mentioned before, our framework can also handle the imperfectly
collateralized contract, where there remain counterparty credit risk as well as collateral
cost adjustment. In the remainder of this chapter, we study several important examples
of imperfect collateralization by using the generic results of Eqs. (3.3.9) and (3.3.10).

**Case 1:** Consider the situation where the both parties use collateral currency $(i)$, which
is the same as the deal currency. In this case, CCA and CVA are given by

$$
CCA_t = E^{Q(i)} \left[ \int_t^T e^{-f^*_s c^{(i)}_s du} y^{(i)}_s \left( - (\delta^1_s - 1)[-\overline{V}_s]^+ + (\delta^2_s - 1)[\overline{V}_s]^+ \right) ds \bigg| \mathcal{F}_t \right] \tag{3.8.1}
$$

and

$$
CVA_t = E^Q \left[ \int_t^T e^{-f^*_s c^{(i)}_s du} h^1_s (1 - R^1_s) \left\{ (1 - \delta^1_s)^+ [-\overline{V}_s]^+ + (\delta^2_s - 1)^+ [\overline{V}_s]^+ \right\} ds \bigg| \mathcal{F}_t \right] 
- E^Q \left[ \int_t^T e^{-f^*_s c^{(i)}_s du} h^2_s (1 - R^2_s) \left\{ (1 - \delta^2_s)^+ [\overline{V}_s]^+ + (\delta^1_s - 1)^+ [-\overline{V}_s]^+ \right\} ds \bigg| \mathcal{F}_t \right] \tag{3.8.2}
$$

where

$$
\overline{V}_t = E^{Q(i)} \left[ \int_{[t,T]} \exp \left( - \int_t^s c^{(i)}_u du \right) dD_s \bigg| \mathcal{F}_t \right] \tag{3.8.3}
$$

7 We assume perfect collateralization just for clearer interpretation. The results will not change qual-
itiatively as long as $\delta^2 y^i > (1-R^1_s)(1-\delta^1_s) h^1_s - (1-R^2_s)(\delta^1_s - 1)^+ h^2_s$. 

is a value under perfect collateralization by domestic currency. This would be the most common situation in the market. Note that the discounting rate is given by the collateral rate and there also exists CCA term.

**Case 2:** Consider the situation where the both parties have to use collateral currency \((j)\) which is different from the deal currency \((i)\). In this case, CCA and CVA are given by

\[
CCA_t = \mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s \left( c_u^{(i)} + y_u^{(i,j)} \right) du} y_s^{(j)} \left( -\left( \delta_s - 1 \right) \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \left( \delta^2_s - 1 \right) \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T \tag{3.8.4}
\]

and

\[
CVA_t = \mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s \left( c_u^{(i,j)} + y_u^{(i,j)} \right) du} h_s^{(j)} \left( 1 - R_{s}^{j} \right) \left( 1 - \delta_s^{1} \right) \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \left( \delta^2_s - 1 \right) \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T
\]

\[
- \mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s \left( c_u^{(i)} + y_u^{(i,j)} \right) du} h_s^{(j)} \left( 1 - R_{s}^{j} \right) \left( 1 - \delta_s^{2} \right) \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \left( \delta^1_s - 1 \right) \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T \tag{3.8.5}
\]

where

\[
\bar{V}_t = \mathbb{E}^{Q(i)} \left[ \int_{[t,T]} \exp \left( - \int_t^s \left( c_u^{(i)} + y_u^{(i,j)} \right) du \right) d\mathcal{D}_s \right] \mathcal{F}_T \tag{3.8.6}
\]

is a value under perfect collateralization by the foreign currency. This is also a common situation for multi- or non-G5 currency trades collateralized by USD but with sizable uncollateralized exposure due to price disputes, for example.

Note that the correlation between the currency funding spread \(y^{(i,j)}\) and hazard rates may contribute significantly to the value of CVA. This is easy to understand, for example, by considering the USD collateralized EUR derivatives with an European bank as a counterparty. As clearly seen in the ongoing turmoil of Euro zone, expensive funding cost of USD reflected by widening EUR/USD CCS basis spread seems highly correlated to the deteriorating creditworthiness of European banks.

**Case 3:** Let us consider another important situation, which is the unilateral collateralization with bilateral default risk. Suppose the setup in **Case 1** except that only the party 1 has to post the collateral due to its low creditworthiness relative to the party 2. In this case, we have

\[
CCA_t = -\mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s c_u^{(i)} du} y_s^{(i)} \left( \delta_s^{1} - 1 \right) \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T \tag{3.8.7}
\]

and

\[
CVA_t = \mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s c_u^{(i,j)} du} h_s^{(j)} \left( 1 - R_{s}^{j} \right) \left( 1 - \delta_s^{1} \right) \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T
\]

\[
- \mathbb{E}^{Q(i)} \left[ \int_t^T e^{-\int_t^s c_u^{(i)} du} h_s^{(j)} \left( 1 - R_{s}^{j} \right) \left( 1 - \delta_s^{2} \right) \begin{bmatrix} \bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} + \begin{bmatrix} -\bar{V}_s^- \\ \bar{V}_s^+ \end{bmatrix} \right] ds \right] \mathcal{F}_T \tag{3.8.8}
\]
where \( V \) is the same as Eq. (3.8.3). If party 1 is required to post "strong" currency (that is the currency with high value of \( y^{(i)} \), such as USD [12]), and also imposed stringent collateral management \( \delta^1 \simeq 1 \), it may suffer significant funding cost from CCA even when CVA is small enough due to the high credit worthiness of the party 2.

Note that, this example is particularly common when SSA (sovereign, supranational and agency) is involved as party 2. For example, when it is a central bank, it does not post collateral but receives it. For the party 1, it is very difficult to hedge this position. Typically, the risk associated with the funding cost in CCA remains un-hedged, since party 1 has to follow bilateral collateralization when it enters an offsetting trade in the interbank market. Once the market interest rate starts to go up while the overnight rate \( c \) is kept low by the central bank, the resultant mark-to-market loss from CCA can be quite significant due to the rising cost of collateral "\( y \)".

Case 4: Our framework can also handle the trades with non-zero collateral thresholds, where margin call occurs only when the exposure to the counterparty exceeds the threshold. A threshold is a level of exposure below which collateral will not be called, and hence it represents the amount of uncollateralized exposure. If the exposure is above the threshold, only the incremental exposure will be collateralized.

Usually, the collateral thresholds are set according to the credit standing of each counterparty. They are often asymmetric, with lower-rated counterparty having a lower threshold than the other. It may be adjusted during the contract according to the "triggers" linked to the credit ratings of the contracting parties. We assume that the threshold of counterparty \( i \) at time \( t \) is set by \( \Gamma^1_i > 0 \), and that the exceeding exposure is perfectly collateralized continuously.

In the presence of thresholds, Eq. (3.2.3) is modified in the following way:

\[
S_t = \beta_t E^Q \left[ \int_{t,T} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \{dD_u + g(u, S_u)S_u du\} \right] + \int_{t,T} \beta_u^{-1} \mathbf{1}_{\{\tau \geq u\}} \{Z^1(u, S_u-)dH_u^1 + Z^2(u, S_u-)dH_u^2 \} \mathcal{F}_t , \tag{3.8.9}
\]

where

\[
g(t, S_t) = y^1_t \left( 1 + \frac{\Gamma^1_t}{S_t} \right) \mathbf{1}_{\{S_t < -\Gamma^1_t\}} + y^2_t \left( 1 - \frac{\Gamma^2_t}{S_t} \right) \mathbf{1}_{\{S_t \geq \Gamma^2_t\}} \tag{3.8.10}
\]

and

\[
Z^1(t, S_t) = S_t \left[ \left( 1 + (1 - R^1_t) \frac{\Gamma^1_t}{S_t} \right) \mathbf{1}_{\{S_t < -\Gamma^1_t\}} + R^1_t \mathbf{1}_{\{-\Gamma^1_t \leq S_t < 0\}} \right] + \mathbf{1}_{\{S_t \geq 0\}}
\]

\[
Z^2(t, S_t) = S_t \left[ \left( 1 - (1 - R^2_t) \frac{\Gamma^2_t}{S_t} \right) \mathbf{1}_{\{S_t \geq \Gamma^2_t\}} + R^2_t \mathbf{1}_{\{0 \leq S_t < \Gamma^2_t\}} \right] + \mathbf{1}_{\{S_t < 0\}}.
\]

Here, we have assumed the same recovery rate for the uncollateralized exposure regardless of whether the contract value is above or below the threshold.

*RRecently, there appear some movements among central banks of European countries toward the 2-way CSA agreements with private financial firms. It seems that these movements are driven by higher fee required from the counterparty when they insist on the 1-way CSA.*
Following the same procedures given in Proposition 1, one can show that the predefined value of the contract \( V_t \) is given by

\[
V_t = E^Q \left[ \int_{[t,T]} \exp \left( -\int_t^s (r_u - \mu(u,V_u)) du \right) dD_s \bigg| F_t \right], \quad t \leq T
\]  

(3.8.11)

where

\[
\mu(t, V_t) = y_t^i \mathbf{1}_{\{V_t < 0\}} + y_t^2 \mathbf{1}_{\{V_t \geq 0\}}
\]

\[
- \left( y_t^1 + h_t^1(1 - R_t^1) \right) \left[ \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < 0\}} - \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} \right]
\]

\[
- \left( y_t^2 + h_t^2(1 - R_t^2) \right) \left[ \mathbf{1}_{\{0 \leq V_t < \Gamma_t^2\}} + \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right].
\]  

(3.8.12)

Now, consider the case where the both parties use the same collateral currency \( \pi \), which is equal to the deal currency. Then, we have

\[
\mu(t, V_t) = y_t^{(i)} - \left\{ y_t^{(i)} \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < \Gamma_t^2\}} \right\}
\]

\[
+ y_t^{(i)} \left[ \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} - \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right]
\]

\[
- h_t^1(1 - R_t^1) \left[ \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < 0\}} - \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} \right]
\]

\[
- h_t^2(1 - R_t^2) \left[ \mathbf{1}_{\{0 \leq V_t < \Gamma_t^2\}} + \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right].
\]  

(3.8.13)

Thus, applying Gateaux derivative around the symmetric perfect collateralization with currency \( \pi \) i.e. \( \overline{y} = y^{(i)} \), we obtain

\[
V_t \simeq \overline{V}_t + \text{CCA} + \text{CVA},
\]  

(3.8.14)

where

\[
\overline{V}_t = E^Q[i] \left[ \int_{[t,T]} \exp \left( -\int_t^s c_u^{(i)} du \right) dD_s \bigg| F_t \right],
\]  

(3.8.15)

and

\[
\text{CCA}_t = -E^Q[i] \left[ \int_t^T e^{- \int_t^s c_u^{(i)} du} y_s^{(i)} \overline{V}_s \mathbf{1}_{\{-\Gamma_s \leq \overline{V}_s < \Gamma_s^2\}} ds \bigg| F_t \right]
\]

\[
+ E^Q[i] \left[ \int_t^T e^{- \int_t^s c_u^{(i)} du} y_s^{(i)} \left[ \Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}} - \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s \geq \Gamma_s^2\}} \right] ds \bigg| F_t \right]
\]  

(3.8.16)

\[
\text{CVA}_t = -E^Q[i] \left[ \int_t^T e^{- \int_t^s c_u^{(i)} du} \left[ h_s^1(1 - R_s^1) \mathbf{1}_{\{-\Gamma_s \leq \overline{V}_s < 0\}} - \Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}} \right] ds \bigg| F_t \right]
\]

\[
- E^Q[i] \left[ \int_t^T e^{- \int_t^s c_u^{(i)} du} \left[ h_s^2(1 - R_s^2) \mathbf{1}_{\{0 \leq \overline{V}_s < \Gamma_s^2\}} + \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s \geq \Gamma_s^2\}} \right] ds \bigg| F_t \right].
\]  

(3.8.17)

It is easy to see that the terms in CCA are reflecting the fact that no collateral is being posted in the range \( \{-\Gamma_t^1 \leq V_t \leq \Gamma_t^2\} \), and that the posted amount of collateral is smaller than \( |V| \) by the size of threshold. The terms in CVA represent bilateral uncollateralized credit exposure, which is capped by each threshold.
3.9 Remarks on the collateral devaluation

Although we have treated $(\delta_i^t)$ as the process of the collateral coverage ratio, it is also useful to handle the collateral devaluation. It is plausible that the value of collateral is highly linked to the counterparty and its value may jump downward at the time of default. For example, we can consider a USD interest rate swap collateralized by EUR cash with an European bank as a counterparty. It is easy to imagine that EUR/USD jump downward at the time of default of the European bank. If we assume the bilateral perfect collateralization, then CCA is zero, but there appears non-zero CVA which can be calculated by interpreting $\delta_i^\tau$ as the fraction of devaluation of the collateral posted by party $i$. We can introduce the collateral coverage ratio and the fraction of collateral devaluation separately to handle more generic situations.

3.10 Conclusions

This chapter develops the methodology to deal with asymmetric and imperfect collateralization as well as remaining counterparty credit risk. It was shown that all of the issues are able to be handled in an unified way by making use of Gateaux derivative. We have shown that the resulting formula contains CCA that represents adjustment of collateral cost due to the deviation from the perfect collateralization, and the terms corresponding to the bilateral CVA. The credit value adjustment now contains the possible dependency among cost of collaterals, hazard rates, collateral coverage ratio and the underlying contract value. Even if we assume that the collateral coverage ratio and recovery rate are constant, the change of effective discounting rate induced by the currency funding spread and its correlation to the hazard rates may significantly change the size of the adjustment.

Direct link of CCS spread and collateral cost allows us to study the numerical significance of asymmetric collateralization. From the numerical analysis using CCS and OIS, the relevance of sophisticated collateral management is now clear. If a financial firm is incapable of posting the cheapest collateral currency, it has to pay very expensive funding cost to the counterparty. We also explained the issue of one-way CSA, which is common when SSA entities are involved. If the funding cost of collateral (or "$y$") rises, the financial firm that is the counterparty of SSA may suffer from significant mark-to-market loss from CCA, and it is quite difficult to hedge.

The article also discussed some generic implications of collateralization. In particular, it was shown that the sophisticated financial firms have an incentive to avoid netting of trades if they try to exploit funding benefit as much as possible, which may reduce the overall netting opportunity and potentially increase the systemic risk in the financial market.
Chapter 4

Collateralized CDS

4.1 Introduction

The last financial crisis has brought about serious research activity on the credit derivatives and counterparty default risk, and large amount of research papers have been published since then. The regulators have also been working hard to establish the new rules for the counterparty risk management, and also for the migration toward the CCPs (central counter parties) particularly in the CDS (credit default swap) market. The excellent reviews and collections of recent works are available in the books edited by Lipton & Rennie (2011) [22] and Bielecki, Brigo & Patras (2011) [2], for example. However, the effects of collateralization on credit derivatives remain still largely unclear. It seems partly because that the idea of collateral cost appeared only recently, and also because the detailed collateral modeling is very complicated, due to the existence of settlement lag, threshold, and minimum transfer amount, e.t.c..

In this chapter, based on the market development toward more stringent collateral management requiring a daily (or even intra-day) margin call, we have studied the pricing of CDS under the assumption of continuous collateralization. This is expected to be particularly relevant for the CCPs dealing with CDS and other credit linked products, for which the assumption of continuous collateralization seems to be a reasonable proxy of the reality. Although we have studied the similar issues for the standard fixed income derivatives in the previous chapter, the result cannot be directly applied to the credit derivatives since the behavior of hazard rates generally violates the so called ”no-jump” condition (e.g. Collin-Dufresne, Goldstein & Hugonnier (2004) [5]) if there exists non-trivial default dependence among the relevant parties \(^1\). In this chapter, we apply the technique introduced by Schönbucher (2000) [27] and adopted later by [5] in order to eliminate the necessity of this condition.

As a result, we have obtained a simple pricing formula for the collateralized CDS. We will see, under the perfect collateralization, that the CDS price does not depend on the counterparty hazard rates at all as expected. However, very interestingly, there remains irremovable trace of the two counter parties through the default dependence. This fact

\(^1\)As mentioned in the remark of Appendix A.1, it is possible to work in the framework of Chapter 3 with careful interpretation of default information sensitive variables, such as hazard rates. The use of ”survival measure” allows systematic treatment which makes the problem more tractable in the presence of non-trivial default dependence.
poses a very difficult problem about the appropriate pricing method for CCPs. A CCP acts as a buyer as well as a seller of a given CDS at the same time, by entering a back-to-back trade between the two financial firms. However, the result tells us that the mark-to-market values of the two offsetting CDSs are not equal in general even at the time of inception, if the CCP adopts the same premium rate for the two firms. Although the detailed work will be left in future works, we think that the result has important implications for the proper operations of CCPs for credit derivatives.

4.2 Fundamental Pricing Formula

4.2.1 Setup

We consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, Q)\), where \(Q\) is a spot martingale measure, and \(\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}\) is a filtration of \(\mathcal{F}\) satisfying the usual conditions. We denote the set of relevant firms \(C = \{0, 1, 2, \ldots, n\}\) and introduce a strictly positive random variable \(\tau_i\) in the probability space as the default time of each party \(i \in C\). We define the default indicator process of each party as \(H^i_t = 1_{\{\tau_i \leq t\}}\) and denote by \(\mathbb{H}^i\) the filtration generated by this process. We assume that we are given a background filtration \(\mathbb{G}\) containing other information except defaults and write \(\mathbb{F} = \mathbb{G} \vee \mathbb{H}^0 \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n\). Thus, it is clear that \(\tau_i\) is an \(\mathbb{H}^i\) as well as \(\mathbb{F}\) stopping time. We assume the existence of non-negative hazard rate process \(h^i\) where

\[
M^i_t = H^i_t - \int_0^t h^i_s 1_{\{\tau^i_s > s\}} ds, \quad t \geq 0 \tag{4.2.1}
\]

is an \((Q, \mathbb{F})\)-martingale. We also assume that there is no simultaneous default for simplicity.

For collateralization, we assume the same setup adopted in Chapter 3 and repeat it here once again for convenience: Consider a trade between the party 1 and 2. If the party \(i (\in \{1, 2\})\) has a negative mark-to-market value, it has to post the cash collateral \(^2\) to the counterparty \(j (\neq i)\), where the coverage ratio of the exposure is denoted by \(\delta^i_t \in \mathbb{R}^+_0\). We assume the margin call and settlement occur instantly. Party \(j\) is then a collateral receiver and has to pay collateral rate \(c^i_t\) on the posted amount of collateral, which is \(\delta^i_t \times (|\text{mark-to-market}|)\), to the party \(i\). This is done continuously until the end of the contract. Following the market conventions, we set the collateral rate \(c^i_t\) as the time-\(t\) value of overnight (O/N) rate of the collateral currency used by the party \(i\). It is not equal to the risk-free rate \(r_t\), in general, which is necessary to make the system consistent with the cross currency market \(^3\). We denote the recovery rate of the party \(i\) by \(R^i_t \in [0, 1]\). We assume that all the processes except default times, such as \(\{c^i, r, \delta^i, R^i\}\) are adapted to the background filtration \(\mathbb{G}\). As for the details of exposure to the counterparty and the recovery scheme, see the corresponding part of Chapter 3.

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\(^2\)According to the ISDA survey \([18]\), more than 80% of collateral being used is cash. If there is a liquid repo or security-lending market, we may also carry out similar formulation with proper adjustments of its funding cost.

\(^3\)See Ref. \([12]\) for details.
4.2.2 CDS Pricing

We denote the CDS reference name by party-0, the investor by party-1, and the counterparty by party-2, respectively. Let us define \( \tau = \tau_0^1 \land \tau_1^1 \land \tau_1^2 \) and its corresponding indicator process, \( H_t = 1_{\{\tau \leq t\}} \). We assume that the investor is a protection buyer and party-2 is a seller. Under this setup, the CDS price from the viewpoint of the investor can be written as

\[
S_t = \beta_t E^Q \left[ \int_{[t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \left( dD_u + q(u, S_u) S_u du \right) \right. \\
+ \left. \int_{[t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \left( Z_u^0 dH_u^0 + Z^1(u, S_u-) dH_u^1 + Z^2(u, S_u-) dH_u^2 \right) \right| F_t \right] \tag{4.2.2}
\]

where \( D \) denotes the cumulative dividend process representing the premium payment for the CDS, and \( \beta_t = \exp \left( \int_0^t r_u ds \right) \) denotes the money-market account with the risk-free interest rate. Other variables are defined as follows:

\[
q(t, v) = \delta_1^1 y_t^1 \mathbf{1}_{\{v < 0\}} + \delta_2^2 y_t^2 \mathbf{1}_{\{v \geq 0\}} \\
Z^0_t = (1 - R^0_t) \tag{4.2.3}
\]

\[
Z^1(t, v) = \left( 1 - (1 - R^1_t) (1 - \delta_1^1)^+ \right) v \mathbf{1}_{\{v < 0\}} + \left( 1 + (1 - R^1_t) (\delta_2^1 - 1)^+ \right) v \mathbf{1}_{\{v \geq 0\}} \\
Z^2(t, v) = \left( 1 - (1 - R^2_t) (1 - \delta_1^2)^+ \right) v \mathbf{1}_{\{v < 0\}} + \left( 1 + (1 - R^2_t) (\delta_2^2 - 1)^+ \right) v \mathbf{1}_{\{v \geq 0\}} .
\]

Here, \( y_t^i = r^i_t - c^i_t \) is the difference of the risk-free and collateral rates relevant for the collateral currency chosen by the party-\( i \) at the time \( t \), which represents the instantaneous return of the posted collateral. Thus the term \( q(t, v) \) summarizes the return (or cost) of collateral from the viewpoint of the investor. \( Z^i \) represents the default payoff when party-\( i \) defaults first among the set \( \{0, 1, 2\} \).

Although we can follow the same procedures used in Chapter 3, we need a careful treatment to avoid the jump in the value process at the time of counterparty default \(^4\).

In this work, we apply the measure change technique introduced by Schönbucher [27] and used by Collin-Dufresne et.al. (2004) [5], which leads to the following proposition in a clearcut way.

**Proposition 4** Under the assumptions given in 4.2.1 and appropriate integrability conditions, the pre-default value \( V_t \) corresponding to the CDS contract specified in Eq. (4.2.2) is given by

\[
V_t = E^Q \left[ \int_{[t,T]} \exp \left( - \int_t^s (r_u - \mu(u, V_u) + h_u^0) du \right) \left( dD_u + Z_u^0 h_u^0 ds \right) \right| F_t \right] \tag{4.2.3}
\]

where

\[
\mu(u, v) = \left( y_t^1 \delta_1^1 -(1 - R^1_t)(1 - \delta_1^1)^+ h_t^1 + (1 - R^2_t)(\delta_1^2 - 1)^+ h_t^2 \right) \mathbf{1}_{\{v < 0\}} \\
+ \left( y_t^2 \delta_2^2 -(1 - R^2_t)(1 - \delta_2^2)^+ h_t^2 + (1 - R^1_t)(\delta_2^2 - 1)^+ h_t^1 \right) \mathbf{1}_{\{v \geq 0\}} \tag{4.2.4}
\]

\(^4\) See the remark given in Appendix A.1.
and it satisfies $S_t = \mathbf{1}_{\{\tau > t\}} V_t$ for all $t \geq 0$. Here, the "survival measure" $Q'$ is defined by

$$\frac{dQ'}{dQ} \bigg|_{\mathcal{F}_t} = \prod_{i=0}^{2} \mathbf{1}_{\{\tau > t\}} \exp \left( \int_0^t \sum_{i=0}^{2} h_i s \, ds \right)$$

(4.2.5)

and the filtration $\mathbb{F}' = (\mathcal{F}'_t)_{t \geq 0}$ denotes the augmentation of $\mathbb{F}$ under $Q'$.

**Proof:** Using the Doob-Meyer decomposition, one obtains

$$S_t = \beta_t E^Q \left[ \int_{[t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} (dD_u + q(u, S_u) S_u \, du) \right.
+ \int_T^t \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \left( Z_0^0 h_u^0 + Z_1^1 (u, S_u) h_u^1 + Z_2^2 (u, S_u) h_u^2 \right) \, du \bigg| \mathcal{F}_t \right].$$

(4.2.6)

Let us define

$$\eta_t = \frac{dQ'}{dQ} \bigg|_{\mathcal{F}_t} = \mathbf{1}_{\{\tau > t\}} \Lambda_t$$

(4.2.7)

where we have used

$$\Lambda_t = \exp \left( \int_0^t \hat{h}_s \, ds \right)$$

(4.2.8)

and $\hat{h}_s = \sum_{i=0}^{2} h_i$. Then, we can proceed as

$$S_t = \beta_t E^Q \left[ \int_{[t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \eta_u (dD_u + q(u, S_u) S_u \, du) \right.
+ \int_T^t \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \left( Z_0^0 h_u^0 + Z_1^1 (u, S_u) h_u^1 + Z_2^2 (u, S_u) h_u^2 \right) \, du \bigg| \mathcal{F}_t \right]
= \mathbf{1}_{\{\tau > t\}} E^Q \left[ \int_{[t,T]} \beta_t \Lambda_t \left\{ dD_u + \left( q(u, V_u) V_u + Z_0^0 h_u^0 + Z_1^1 (u, V_u) h_u^1 + Z_2^2 (u, V_u) h_u^2 \right) \, du \right\} \bigg| \mathcal{F}'_t \right].$$

Thus, on the set $\{\tau > t\}$, we can write

$$V_t = E^Q \left[ \int_{[t,T]} \beta_t \Lambda_t \left\{ dD_u + \left( q(u, V_u) V_u + Z_0^0 h_u^0 + Z_1^1 (u, V_u) h_u^1 + Z_2^2 (u, V_u) h_u^2 \right) \, du \right\} \bigg| \mathcal{F}'_t \right].$$

Simple algebra gives us

$$V_t = E^Q \left[ \int_{[t,T]} e^{- \int_t^r (\tau_u + \hat{h}_u) \, du} \left\{ dD_s + Z_0^0 h_s^0 \, ds + \left( \hat{h}_s - h_s^0 + \mu(s, V_s) \right) V_s \, ds \right\} \bigg| \mathcal{F}'_t \right].$$

Integrating the linear terms gives us the desired result. □
4.3 Financial Implications

Following the approximation method in Chapter 3, we can derive price adjustments in various situations. However, in the reminder of the chapter, let us concentrate on the simplest but important situation where the CDS is perfectly collateralized by the domestic currency, which should be also the most relevant case for the operations of CCPs. In this case, we have \( \delta^1 = \delta^2 = 1 \) and \( y^1 = y^2 = r - c \). Thus, using the result of Proposition 4, the pre-default value of the CDS is given by

\[
V_t = E^{Q'} \left[ \int_{\beta,T} \exp \left( - \int_t^s (c_u + h_u^0) du \right) \left( dD_s + Z_s h_s^0 ds \right) \bigg| \mathcal{F}_t' \right]
\]

and one can easily confirm that the hazard rates of the investor 1 as well as the counterparty 2 are absent from the pricing formula. Naively, it looks as if we succeed to recover the risk-free situation by the stringent collateral management. However, we will just see that it is quite misleading and dangerous to treat the result of Eq. (4.3.1) as the usual risk-free pricing formula.

The key point resides in the new measure \( Q' \) and the filtration \( \mathcal{F}' \). As was emphasized in the works [27, 5], the transformation in Eq. (4.2.5) puts zero weight on the events where the parties \( \{0, 1, 2\} \) default. It can be easily checked as follows: By construction, we know that

\[
M_t = H_t - \int_0^t (1 - H_s) \tilde{h}_s ds
\]

is a \((Q, \mathcal{F})\)-martingale. Then, Maruyama-Girsanov’s theorem implies that

\[
M'_t = M_t - \int_0^t \frac{d\langle M, \eta \rangle_s}{\eta_s}
\]

should be a \((Q', \mathcal{F}')\)-martingale, where \( \langle \cdot, \cdot \rangle \) denotes the (conditional or predictable) quadratic covariation. Now, one can easily check that

\[
M'_t = M_t + \int_0^t (1 - H_s) \tilde{h}_s ds = H_t
\]

and thus \( H_t = 1_{\{\tau \leq t\}} \) itself becomes a \((Q', \mathcal{F}')\)-martingale. In other words, under the new measure, the parties \{0, 1, 2\} do not default almost surely. Since the density process in (4.2.7) has finite variation, there is no change of dynamics in the continuous part.

Let us consider the financial implications of this fact. By our construction of filtration, \((Q, \mathcal{F})\) hazard rate process of party \( i \) can be written in the following form in general:

\[
h_i^D(t) = \sum_{\{D \in \Pi: \forall i \in D\}} \prod_{J \in D} 1_{\{\tau_J \leq t\}} \prod_{k \in \mathcal{C} \setminus D} 1_{\{\tau_k > t\}} h_D(t)
\]
although they are not explicitly shown in the formula. Importantly, it is not the hazard rate (or default intensity) under the new measure ($Q', F'$).

In the new measure, the parties included in $S = \{0, 1, 2\}$ are almost surely alive for all $t \geq 0$, and hence $h^i_t$ given in Eq. (4.3.5) is equivalent to $h'^i_t$ given below:

$$h'^i_t = \sum_{\{D \in \Pi'; i \notin D\}} \prod_{j \in D} \prod_{k \in C' \setminus D} \left(1_{\{\tau^j \leq t\}} \prod_{k \in C' \setminus D} 1_{\{\tau^k > t\}} h^i_D(t) \right)$$  \hspace{1cm} (4.3.6)

where $C' = C \setminus S$, and $\Pi'$ denotes the set of all the subgroups of $C'$ and the empty set. Thus, the pre-default value of the CDS can also be written as

$$V_t = E^{Q'} \left[ \int_{[t,T]} \exp \left( - \int_t^s (c_u + h'^0_u)du \right) \left( dD_s + Z^0_s h'^0_s ds \right) \bigg| F'_t \right].$$ \hspace{1cm} (4.3.7)

Now, let us consider the difference from the situation where the same CDS is being traded between the two completely default-free parties. In this case, the pre-default value of the CDS is given by

$$V_{rf} = E^{Q} \left[ \int_{[t,T]} \exp \left( - \int_t^s (c_u + h^0_u)du \right) \left( dD_s + Z^0_s h^0_s ds \right) \bigg| F_t \right].$$ \hspace{1cm} (4.3.8)

Here, we have assumed that the collateralization is still being carried out. In the reminder of the paper, we study the important difference

$$V_{rf} - V_t$$ \hspace{1cm} (4.3.9)

in details.

Here, let us briefly discuss the cause of the above difference intuitively. As one can see from (4.3.5), the hazard rate in the full filtration $F$ jumps at the default of any firm if there is non-zero default dependence with the reference firm $^5$. Suppose the firm $A$ has positive dependence to the reference name. Then, the value of CDS protection bought from the default-free firm jumps upward when $A$ defaults and the $V_{rf}$ contains the discounted contribution from this scenario. Now, suppose the situation where we bought a protection from the firm $A$. In this case, the contribution from the future scenarios where $A$ defaults before the contract expires is smaller than that in the previous situation. This is because the collateral posted by the firm $A$ cannot cover the upward jump of CDS at the time of default. The similar effect also arises from the investor himself. As a result, if there is positive (or contagious) default dependence among the participants and the reference name, the CDS value is expected to be smaller than $V_{rf}$.

The intuitive interpretation of the result is rather difficult, but it is possible to consider the result $V_t$ as a "clean" price of the contract if we correctly understand the CDS as a protection against the default of the reference name on the condition that the investor as well as the counterparty do not default by the maturity or the default of the reference name. A continuously collateralized CDS among the risky parties $\{0, 1, 2\}$ has the same price for that of the same reference name between the two default-free entities such as central banks with the above described condition. Although it is difficult to calculate this difference in generic setup, we will provide the explicit formula in the two simplified but important situations in the following sections.

\footnote{See the discussion in Sec. 4.5.}
4.4 Special Cases

We first list up the two special cases which provide us better understanding. Especially, the second example can be applied to the back-to-back trade, which is the most relevant situation for the CCPs.

4.4.1 3-party Case

The simplest situation to calculate the collateralized CDS is the case where there are only three relevant names, $\mathcal{C} = \{0, 1, 2\}$, which are the reference entity, the investor and the counterparty, respectively. In this case, since $\mathcal{S} = \mathcal{C}$, the set $\mathcal{I}^{0}$ contains only the empty set $\{\emptyset\}$. Thus, under the survival measure $(Q', \mathcal{F}')$, we have

$$h_{t}^{i} = h_{\{i\}}^{0}(t) \quad (4.4.1)$$

and particularly,

$$h_{t}^{0} = h_{\{0\}}^{0}(t). \quad (4.4.2)$$

Since we know that $h_{\{i\}}^{j}$ is adapted to the background filtration $\mathcal{G}$, the evaluation of CDS value is quite straightforward in this case.

4.4.2 4-party Case

Now, let us add one more party and consider the 4-party case, $\mathcal{C} = \{0, 1, 2, 3\}$. We first consider the trade of a CDS between the investor (party-1) and the counterparty (party-2). The reference entity is party-0. In this case, we have $\mathcal{C}' = \{3\}$. Therefore, the relevant processes under the survival measure are

$$h_{t}^{0} = 1_{\{\tau^{3} > t\}}h_{\{0\}}^{0}(t) + 1_{\{\tau^{3} \leq t\}}h_{\{3\}}^{0}(t, \tau^{3}) \quad (4.4.3)$$

$$h_{t}^{3} = 1_{\{\tau^{3} > t\}}h_{\{3\}}^{0}(t). \quad (4.4.4)$$

Here, we have made the dependence on the default time in $h_{\{3\}}^{0}$ explicitly. Since the feedback effect only appears through the party 3 in the survival measure, we can proceed in similar fashion done in Ref. [5].

In this case, the perfectly collateralized CDS pre-default value turns out to be

$$V_{t} = 1_{\{\tau^{3} \leq t\}}E^{Q'} \left[ \int_{[t,T]} e^{-\int_{u}^{c} (c_{a} + h_{\{3\}}^{3}(u, \tau^{3})) du} \left( dD_{s} + Z_{s}^{0} h_{\{3\}}^{0}(s, \tau^{3}) ds \right) \bigg| \mathcal{F}_{t} \right]$$

$$+ 1_{\{\tau^{3} > t\}} \left\{ E^{Q'} \left[ \int_{[t,T]} e^{-\int_{u}^{c} (c_{a} + h_{\{3\}}^{3}(u)) du} \left( e^{-\int_{u}^{c} h_{\{3\}}^{3}(u) du} \right) \left( dD_{s} + Z_{s}^{0} h_{\{3\}}^{0}(s) ds \right) \bigg| \mathcal{F}_{t} \right] \right\}$$

$$+ E^{Q'} \left[ \int_{[t,T]} e^{-\int_{u}^{c} c_{a} du} \left( \int_{t}^{s} e^{-\int_{v}^{c} h_{\{0\}}^{3}(u) du} \left( \int_{t}^{s} e^{-\int_{v}^{c} h_{\{0\}}^{3}(u, \tau^{3}) du} \bigg| \mathcal{F}_{t} \right) \right) \right]$$

$$+ E^{Q'} \left[ \int_{[t,T]} e^{-\int_{u}^{c} c_{a} du} \left( \int_{t}^{s} e^{-\int_{v}^{c} \left( f_{n}^{3} h_{\{0\}}^{3}(u) + f_{n}^{3} h_{\{3\}}^{3}(u, \tau^{3}) \right) du} \left( e^{-\int_{u}^{c} h_{\{0\}}^{3}(x) dx} h_{\{3\}}^{3}(v) \bigg| \mathcal{F}_{t} \right) \right) \right]$$

$$+ E^{Q'} \left[ \int_{[t,T]} e^{-\int_{u}^{c} c_{a} du} \left( \int_{t}^{s} e^{-\int_{v}^{c} \left( f_{n}^{3} h_{\{0\}}^{3}(u) + f_{n}^{3} h_{\{3\}}^{3}(u, \tau^{3}) \right) du} \left( e^{-\int_{u}^{c} h_{\{0\}}^{3}(x) dx} h_{\{3\}}^{3}(v) \bigg| \mathcal{F}_{t} \right) \right) \right] \right\}. \quad (4.4.5)$$
Here, the important point is not the possible contagious effect from the party-3, but rather the lack of the contagious effects from the other names included in the set $S$, which are included in the marginal default intensity of the reference entity.

Now, because of the symmetry, if the investor enters a back-to-back trade with the counterparty 3, the pre-default value of this offsetting contract is given as follows:

$$V_t^{B2B} = -1_{\{\tau^2 \leq t\}} E^{Q''} \left[ \int_{[t,T]} e^{-\int_t^s \left( c_u + h^0_{(2)}(u,t^2) \right) du} \left( dD_s + Z_s^0 h^0_{(2)}(s,s,\tau^2)ds \right) \bigg| \mathcal{F}_t'' \right]$$

$$-1_{\{\tau^2 > t\}} \left\{ E^{Q''} \left[ \int_{[t,T]} e^{-\int_t^s \left( c_u + h^0_{(2)}(u) \right) du} \left( e^{-\int_t^s h^2_{(2)}(u)du} \right) \left( dD_s + Z_s^0 h^0_{(2)}(s)ds \right) \bigg| \mathcal{F}_t'' \right] ight\}$$

$$+ E^{Q''} \left[ \int_{[t,T]} e^{-\int_t^s c_u du} \left( \int_t^s e^{-\left( \int_t^x h^0_{(2)}(u) + \int_t^x h^0_{(2)}(u,v) \right) du} \left( e^{-\int_t^x h^2_{(2)}(x)dx} h^2_{(2)}(v) \right) dv \right) dD_s \bigg| \mathcal{F}_t'' \right]$$

$$+ E^{Q''} \left[ \int_{[t,T]} e^{-\int_t^s c_u du} \left( \int_t^s e^{-\left( \int_t^x h^0_{(2)}(u) + \int_t^x h^0_{(2)}(u,v) \right) du} \left( e^{-\int_t^x h^2_{(2)}(x)dx} h^2_{(2)}(v) \right) h^0_{(2)}(s,v)dv \right) Z_s^0 ds \bigg| \mathcal{F}_t'' \right] \right\}.$$

Here, $Q''$ is defined according to the new survival set $\mathcal{S}^{B2B} = \{0, 1, 3\}$ by

$$\frac{dQ''}{dQ} \bigg|_{\mathcal{F}_t} = \prod_{i \in \mathcal{S}^{B2B}} 1_{\{\tau^2 > t\}} \exp \left( \int_0^t \sum_{i \in \mathcal{S}^{B2B}} h_i^3 ds \right)$$

and the filtration $\mathcal{F}'' = (\mathcal{F}_t'')_{t \geq 0}$ denotes the augmentation of $\mathcal{F}$ under $Q''$.

Now let us consider $V_0 + V_0^{B2B}$. It is easy to check that it is not zero in general and does depend on the default intensities of party-2 and -3, and also their contagious effects to the reference entity. Suppose that the investor is a CCP just entered into the back-to-back trade with the party-2 and -3 who have the same marginal default intensities. Even under the perfect collateralization, if the CCP applies the same CDS price (or premium) to the two parties, it has, in general, the mark-to-market loss or profit even at the inception of the contract. For example, consider the case where the protection seller (party-2) has very high default dependence with the reference entity, while the buyer (party-3) of the protection from the CCP has smaller one. In this case, the $h^3$ in $(Q', \mathcal{F}')$ should be smaller than that of $(Q'', \mathcal{F}'')$. If the CCP uses the same CDS premium to the two parties, CCP should properly recognizes the loss, which stems from the difference of the contagion size from the default of the two parties. Since the party 2 has high default dependence, the short protection position of the CCP with the party 3 suffers bigger loss at the time of the default of the party 2.

### 4.5 Examples using a Copula

In order to separate the marginal intensity and default dependence, we will adopt here the copula framework. After explaining the general setup, we will apply Clayton copula to demonstrate quantitative impact.
4.5.1 Framework

Suppose that we are given a non-negative process \( \lambda^i \) adapted to the background filtration \( \mathbb{G} \) for each party \( i \in C = \{0, 1, 2, \cdots, n\} \). Suppose also that there exists a uniformly distributed random variable \( U^i \in [0, 1] \). We assume that, under \( (Q, \mathcal{F}_0) \), the \((n + 1)\)-dimensional random vector

\[
\vec{U} = (U^0, U^1, \cdots, U^n)
\]

is distributed according to the \((n + 1)\)-dimensional copula

\[
C(\vec{u}) .
\]

We further assume that \( \vec{U} \) is independent from \( \mathbb{G}_\infty \) and also that copula function \( C \) is \((n + 1)\)-times continuously differentiable.

Now, let us define the default time \( \tau^i \) as

\[
\tau^i = \inf \{ t; e^{-\int_0^t \lambda^i_s ds} \leq U^i \} .
\]

Then, given the information \( \mathbb{G}_\infty \), one obtains the joint default distribution as

\[
Q \left( \tau^0 > T^0, \tau^1 > T^1, \cdots, \tau^n > T^n \mid \mathbb{G}_\infty \right) = C(\vec{\gamma}(\vec{T}))
\]

where we have used the notation of

\[
C(\vec{\gamma}(\vec{T})) = C(\gamma^0(T^0), \cdots, \gamma^n(T^n))
\]

and

\[
\gamma^i(T) = \exp \left( - \int_0^T \lambda^i_s ds \right) .
\]

Following the well known procedures \(^6\), one obtains

\[
Q^i(t, T) = Q \left( 1_{\{\tau^i > T^i\}} \mid \mathcal{F}_t \right) = \sum_{\{D \in \Pi; i \notin D\}} \prod_{j \notin D} 1_{\{\tau^j \leq t\}} \prod_{k \in C \setminus D} 1_{\{\tau^k > t\}} \frac{\partial D C(\gamma^i(T^i), \vec{\gamma}^{C \setminus D}(t), \vec{\gamma}^D(\vec{t}))}{\partial D C(\vec{\gamma}^{C \setminus D}(t), \vec{\gamma}^D(\vec{t}))} \bigg|_{\vec{D} = \vec{t}^D}.
\]

Here, we have defined \( \Pi \) as the set containing all the subgroups of \( C \) with the empty set, and

\[
\tilde{\partial}_D = \prod_{i \in D} \frac{\partial}{\partial u^i} .
\]

\( \vec{\gamma}^D(\vec{t}^D) \) is the set of \( \gamma^i(\tau^i) \) for all \( i \in D \), and similarly for \( \vec{\gamma}^{C \setminus D} \). In the expression of Eq. (4.5.7), we have not properly ordered the arguments of the copula function just for simplicity, which should be understood in the appropriate way.

\(^6\)See, the works[28, 26], for example.
\((Q, F)\) hazard rate of party-\(i\) is calculated as

\[
h_i^t = - \frac{\partial}{\partial T} \ln Q^i(t, T) \bigg|_{T=t} = \lambda_i^t \gamma_i(t) \sum_{D \in \Pi} \prod_{j \notin D} 1_{\{r_j \leq t\}} \prod_{k \in C \setminus D} 1_{\{r_k > t\}} \frac{\partial^2 \tilde{D}(\tilde{C}_{(D)}(t), \tilde{D}(\tilde{T}))}{\partial_{\tilde{D}}(\tilde{C}_{(D)}(t), \tilde{D}(\tilde{T}))}. \tag{4.5.9}
\]

Hence, as we have done in Eq (4.3.6) for the survival set \(S = \{0, 1, 2\}\), \(h_i^t\) can be replaced as follows under the new measure \((Q', F')\):

\[
h_i^t = \lambda_i^t \gamma_i(t) \sum_{D \in \Pi'} \prod_{j \notin D} 1_{\{r_j \leq t\}} \prod_{k \in C \setminus D} 1_{\{r_k > t\}} \frac{\partial^2 \tilde{D}(\tilde{C}_{(D)}(t), \tilde{D}(\tilde{T}))}{\partial_{\tilde{D}}(\tilde{C}_{(D)}(t), \tilde{D}(\tilde{T}))} \tag{4.5.10}
\]

where \(C' = C \setminus S\) and \(\Pi'\) is the set of all the subgroups of \(C'\) with an additional empty set. One can observe that both \(h^t\) and \(h'^t\) are equal to the marginal intensity \(\lambda^t\) if there is no default dependence (or in the case of the product copula).

Now, let us apply the copula framework to the two special cases given in Sec. 4.4.

(3-party Case): In the 3-party case with \(C = \{0, 1, 2\}\), we have

\[
h_{i0}^t = \lambda_i^0 \gamma_0(t) \frac{\partial \tilde{C}(\tilde{C}^0(t))}{\tilde{C}(\tilde{C}^0(t))}, \tag{4.5.11}
\]

where \(\tilde{C}(t) = (\gamma^0(t), \gamma^1(t), \gamma^2(t))\).

(4-party Case): In the 4-party case with \(C = \{0, 1, 2, 3\}\), we have

\[
h_{i0}^t = 1_{\{r_1 < t\}} \frac{\partial \tilde{C}(\tilde{C}^0(t))}{\tilde{C}(\tilde{C}^0(t))} + 1_{\{r_1 \leq t\}} \frac{\partial \tilde{C}(\tilde{C}^{[1]}(t))}{\tilde{C}(\tilde{C}^{[1]}(t))} \tag{4.5.12}
\]

\[
h_{i3}^t = 1_{\{r_3 > t\}} \frac{\partial \tilde{C}(\tilde{C}^3(t))}{\tilde{C}(\tilde{C}^3(t))} \tag{4.5.13}
\]

under \((Q', F')\), and

\[
h_{i0}^{r2} = 1_{\{r_2 > t\}} \frac{\partial \tilde{C}(\tilde{C}^0(t))}{\tilde{C}(\tilde{C}^0(t))} + 1_{\{r_2 \leq t\}} \frac{\partial \tilde{C}(\tilde{C}^{[2]}(t))}{\tilde{C}(\tilde{C}^{[2]}(t))} \tag{4.5.14}
\]

\[
h_{i0}^{r0} = 1_{\{r_2 > t\}} \frac{\partial \tilde{C}(\tilde{C}^{[0]}(t))}{\tilde{C}(\tilde{C}^{[0]}(t))} \tag{4.5.15}
\]

under \((Q'', F'')\).

### 4.5.2 Numerical Examples

In this paper, we adopt Clayton copula just for its analytical tractability and easy interpretation of its parameter \([28]\). Clayton copula belongs to Archimedean copula family, whose general form is given by

\[
C(u) = \phi^{-1} \left( \sum_{i=0}^{n} \phi(u_i^i) \right) \tag{4.5.16}
\]
where the function $\phi(\cdot)$ is called the generator of the copula, and $\phi^{-1}(\cdot)$ is its pseudo-inverse function. For Clayton copula, the generator function is given by

$$\phi(u) = (u^{-\alpha} - 1)/\alpha$$

(4.5.17)

for $\alpha > 0$, and hence we have

$$C(\vec{u}) = \left(-n + \sum_{i=0}^{n} (u^i)^{-\alpha}\right)^{-1/\alpha}.$$

(4.5.18)

For this copula, one can easily check that

$$\gamma^i \partial_i C(\vec{\gamma}) = \left(\frac{C(\vec{\gamma})}{\gamma^i}\right)^{\alpha}$$

$$\gamma^i \partial_i \partial_j C(\vec{\gamma}) = (1 + \alpha)\left(\frac{C(\vec{\gamma})}{\gamma^i}\right)^{\alpha}$$

(4.5.19)

which means that the hazard rate jumps to $(1+\alpha)$ times of its value just before the default of any other party.

In the following two figures, we have shown the numerical examples of the par premium of the perfectly collateralized CDS under Clayton copula. For simplicity, we have assumed continuous payment of the premium, and also assumed that the recovery rate $R = 40\%$, the collateral rate $c = 0.02$, and the marginal intensity $\lambda^i$ of each name is constant.

In Fig. 4.1, we have shown the results of 3-party case for the set of maturities; 1yr, 5yr, 10yr and 20yr. Here, the effective marginal intensity of each party $\vec{\lambda} = (1-R^i)\lambda^i$ is given as follows:

$$\left(\lambda^0, \lambda^1, \lambda^2\right) = (200bp, 100bp, 120bp).$$

The horizontal axis denotes the value of the copula parameter $\alpha$. Considering the situation where major financial institutions are involved, and recalling the events just after the Lehman collapse, the significant jump of hazard rates seems possible. One can see that there is meaningful deviation of the par CDS premium from the marginal intensity even within the reasonable range of the jump size, or $\alpha$.

If Fig. 4.2, we have shown the corresponding results for the 4-party case. Here, we have set the effective marginal intensities as

$$\left(\lambda^0, \lambda^1, \lambda^2, \lambda^3\right) = (200bp, 30bp, 150bp, 75bp).$$

In this case, we have modeled the situation where the investor has very high credit quality, which enters back-to-back trades with the two firms that have quite different credit worthiness. In the figure, we have used the solid lines for the trade with party-2, and the dashed lines for the offsetting trade with party-3. The result tells us that the back-to-back trades have non-zero mark-to-market value if the investor applies the same premium. This fact is very important for a CCP. It tells us that, even under the very stringent collateral management, the CCP has to recognize that it is not free from the "risk". It is, in fact, free from the credit risk of the counterparty, but still suffers from the contagious effects from the defaults of its counter parties.
Figure 4.1: Change of par CDS spread with Clayton copula parameter $\alpha$.

Figure 4.2: Change of par CDS spread with Clayton copula parameter $\alpha$. 
Remark on CVA and CCA

It is also straightforward to derive the leading order approximation under the imperfect collateralization, or \(\delta^i \neq 1\), using the technique in Chapter 3. Assume symmetric collateralization with domestic currency, but consider the imperfect collateralization \(\delta^i_t \neq 1\). In this case, in the leading order approximation, we have

\[
V_t \simeq \overline{V}_t + CCA_t + CVA_t
\]  \(4.5.20\)

where

\[
\overline{V}_t = E^{Q} \left[ \int_{[t,T]} \exp \left( -\int_t^s (c_u + h_u^0)du \right) \left( dD_s + Z_s^0 h_s^0 ds \right) \right] F_t'
\]  \(4.5.21\)

which is a perfectly collateralized price and

\[
CCA_t = E^{Q} \left[ \int_t^T e^{-\int_t^s (c_u + h_u^0)du} y_s \left\{ (1 - \delta^1_s)^+ [-\overline{V}_s]^+ - (1 - \delta^2_s)^+ [\overline{V}_s]^+ \right\} ds \right] F_t'
\]  \(4.5.22\)

\[
CVA_t = E^{Q} \left[ \int_t^T e^{-\int_t^s (c_u + h_u^0)du} (1 - R^1_s) h_s^1 \left\{ (1 - \delta^1_s)^+ [-\overline{V}_s]^+ + (\delta^2_s - 1)^+ [\overline{V}_s]^+ \right\} ds \right] F_t'
\]  \(4.5.23\)

Let us consider the four party case treated in Sec. 4.4.2. For the expression of \(\overline{V}_t\), we can use the same result given in Eq. (4.4.5). As for \((h^1, h^2)\) in \((Q', F')\), we need to apply the formula (4.3.6) or (4.5.10). The results can be written similarly as Eq. (4.5.12) in a copula framework, for example. It is clear that the value of CVA (and also CCA) has finite gap before and after the default of names included in the set \(C'\) or \(\{3\}\) (\(\{2\}\) in B2B) in the example.

4.6 Conclusions

In this paper, we have studied the pricing of CDS under continuous collateralization. We have made use of the "survival measure" to avoid the "no-jump" assumption. It allows us straightforward derivation of pricing formula of CDS.

In the main body of the paper, we have focused on the situation where the CDS is perfectly collateralized. We have shown that there exists irremovable trace of the two participating firms in the CDS price through their default dependence with the reference name. For numerical examples, we have adopted Clayton copula to show the change of the par CDS premium according to the dependence parameter. The results have shown that there exists significant deviation of the par premium from the marginal intensity of the reference entity when the default dependence is high. This fact seems particularly important for the CCPs, where the members are usually major broker-dealers that are expected to have very significant impacts on all the other names just we have experienced in this crisis.
Chapter 5

Concluding Remarks

In the thesis, we have presented the pricing framework necessary for the post crisis epoch. In particular, we have explained the benchmark pricing under the full collateralization in details, where the widespread use of collateralization and elevated basis spreads have played an important role. Although the importance of CVA (credit value adjustment) is well recognized among financial firms and regulatory authorities, CVA is nevertheless a "correction" to the benchmark and suffers a significant model dependency and the lack of clean information in the market. In order to have the transparency in pricing which is crucial to achieve enough liquidity in the market, and also to be compliant on the new set of regulations, the current market trend is strongly driven toward fully collateralized contracts either in the OTC or through CCPs. Here, the contents of Chapter 2 are quite likely to be the next market standard. In fact, some of the underlying papers for the chapter are now among the standard references for practitioners.

However, one should recognize that this framework is just an emergency stopgap which is still based on traditional no-arbitrage pricing in the frictionless and essentially complete market. Now, a set of new regulations are going to constrain the financial contracts directly, or indirectly but in a quite effective fashion through those on banks' balance sheets. For example, a requirement to hold greater amount of creditworthy, highly liquid and unencumbered assets combined with almost unavoidable collateralization in the market seems to serve as an effective ceiling on the feasible size of trading books. The research on the pricing and risk-management issues in constrained and incomplete markets is still largely unexplored. One of the biggest obstacles is the non-linearity appearing in the system to be solved, which is often described by non-linear forward backward stochastic differential equations (FBSDEs), or equivalent HJB equations.

We have already seen an example of FBSDE in Chapter 3 dealing with imperfect collateralization, where the first order approximation does a good job in the normal market. However it is unclear for highly stressed environments and for other general problems if the first order approximation is accurate enough. Recently, in Fujii &Takahashi (2011, 2012) [16, 17], we have made some progress on its approximation technique that allows to derive higher order correction terms systematically. Further improvement and its application will be a very hard but worthwhile endeavor to make a long trek, which will lead us to the problems of optimal portfolio, indifference pricing, non-linear filtering and their various combinations.
Appendix A

Appendix for Chapter 2

A.1 Proof of Proposition 1

Firstly, we consider the SDE for $S_t$. Let us define $L_t = 1 - H_t$. One can show that

$$
\beta_t^{-1} S_t + \int_{[0,t]} \beta_u^{-1} L_u (dD_u + q(u, S_u) S_u du) + \int_{[0,t]} \beta_u^{-1} L_u (Z^1(u, S_{u-}) dH^1_u + Z^2(u, S_{u-}) dH^2_u) 
$$

$$
= E^Q \left[ \int_{[0,T]} \beta_u^{-1} 1_{\{t > u\}} \left\{ dD_u + (y_u^1 \delta_u^1 1_{\{S_u < 0\}} + y_u^2 \delta_u^2 1_{\{S_u \geq 0\}}) S_u du \right\} 
+ \int_{[0,T]} \beta_u^{-1} L_u \left( Z^1(u, S_{u-}) dH^1_u + Z^2(u, S_{u-}) dH^2_u \right) \right] = m_t \quad (A.1.1)
$$

where

$$
q(t, v) = y^1_t \delta^1_t 1_{\{v < 0\}} + y^2_t \delta^2_t 1_{\{v \geq 0\}} \quad (A.1.2)
$$

and $\{m_t\}_{t \geq 0}$ is a $Q$-martingale. Thus we obtain the following SDE:

$$
dS_t - r_t S_t dt + L_t (dD_t + q(t, S_t) S_t dt) + L_t (Z^1(t, S_{t-}) dH^1_t + Z^2(t, S_{t-}) dH^2_t) = \beta_t dm_t . \quad (A.1.3)
$$

Using the decomposition of $H^1_t$, we get

$$
dS_t - r_t S_t dt + L_t (dD_t + q(t, S_t) S_t dt) + L_t (Z^1(t, S_t) h^1_t + Z^2(t, S_t) h^2_t) dt = dn_t , \quad (A.1.4)
$$

where we have defined

$$
dn_t = \beta_t dm_t - L_t (Z^1(t, S_{t-}) dM^1_t + Z^2(t, S_{t-}) dM^2_t) \quad (A.1.5)
$$

and $\{n_t\}_{t \geq 0}$ is also a some $Q$-martingale. Using the fact that

$$
q(t, S_t) S_t + Z^1(t, S_t) h^1_t + Z^2(t, S_t) h^2_t = S_t (\mu(t, S_t) + h_t) \quad (A.1.6)
$$

one can show that the SDE for $S_t$ is given by

$$
dS_t = -L_t dD_t + L_t (r_t - \mu(t, S_t) - h_t) S_t dt + dn_t . \quad (A.1.7)
$$
Secondly, let us consider the SDE for $V_t$. By following the similar procedures, one can easily see that
\[
e^{-\int_0^t (r_u - \mu(u,V_u)) \, du} V_t + \int_{[0,t]} e^{-\int_0^s (r_u - \mu(u,V_u)) \, du} \, dD_s
\]
\[
= E^Q \left[ \int_{[0,T]} \exp \left( - \int_0^s (r_u - \mu(u,V_u)) \, du \right) \, dD_u \left| F_t \right. \right] = \tilde{m}_t , \tag{A.1.8}
\]
where $\{\tilde{m}_t\}_{t \geq 0}$ is a Q-martingale. Thus we have
\[
dV_t = -dD_t + \left( r_t - \mu(t,V_t) \right) V_t dt + d\tilde{m}_t , \tag{A.1.9}
\]
where
\[
d\tilde{m}_t = e^{\int_0^t (r_u - \mu(u,V_u)) \, du} \, d\tilde{m}_t , \tag{A.1.10}
\]
and hence $\{\tilde{m}_t\}_{t \geq 0}$ is also a Q-martingale. As a result we have
\[
d(1_{\{\tau > t\}} V_t) = d(L_t V_t)
\]
\[
= L_{t^-} dV_t - V_t^- dH_t - \Delta V_t \Delta H_t
\]
\[
= -L_{t^-} dD_t + L_t \left( r_t - \mu(t,V_t) \right) V_t dt - L_t V_t h_t dt - \Delta V_t \Delta H_t
\]
\[
+ L_{t^-} \left( d\tilde{m}_t - V_t^- (dM_t^1 + dM_t^2) \right)
\]
\[
= -L_t dD_t + L_t \left( r_t - \mu(t,V_t) - h_t \right) V_t dt - \Delta V_t \Delta H_t + d\tilde{N}_t , \tag{A.1.11}
\]
where $\{\tilde{N}_t\}_{t \geq 0}$ is a Q-martingale such that
\[
d\tilde{N}_t = L_{t^-} \left( d\tilde{m}_t - V_t^- (dM_t^1 + dM_t^2) \right) . \tag{A.1.12}
\]
Therefore, by comparing Eqs. (A.1.7) and (A.1.11) and also the fact that $S_T = 1_{\{\tau > T\}} V_T = 0$, we cannot distinguish $1_{\{\tau > t\}} V_t$ from $S_t$ if there is no jump at the time of default $\Delta V_t = 0$. ■

Remark: In this remark, we briefly discuss the assumption of $\Delta V_t = 0$. Notice that, since we assume totally inaccessible default time, there is no contribution from pre-fixed lump-sum coupon payments to the jump. In addition, it is natural (and also common in the existing literatures) to assume global market variables, such as interest rates and FX’s, are adapted to the background filtration independent from the defaults. In this paper, we are concentrating on the standard fixed income derivatives without credit sensitive dividends, and hence the only thing we need to care about is the behavior of hazard rates, $h^1$ and $h^2$. Therefore, in this case, if there is no jump on $h^i$ on the default of the other party $j \neq i$, then the assumption $\Delta V_t = 0$ holds true. This corresponds to the situation where there is no default dependence between the two firms.

If there exists non-zero default dependence, which is important in risk-management point of view, then there appears a jump on the hazard rate of the surviving firm when a default occurs. This represents a direct feedback (or a contagious effect) from the defaulted firm to the surviving one. In this case, if we directly use $\mathcal{F}$-intensities $h^i$, the no-jump assumption does not hold.
However, even in this case, there is a way to handle the pricing problem correctly. Let us construct the filtration in the usual way as $\mathcal{F}_t = \mathcal{G}_t \lor \mathcal{H}_1^t \lor \mathcal{H}_2^t$, where $\mathcal{G}_t$ is the background filtration (say, generated by Brownian motions), and $\mathcal{H}_i^t$ is the filtration generated by $H^i$. Since the only information we need is up to $\tau = \tau_1 \land \tau_2$, we can limit our attention to the intensities conditional on no-default, which are now the processes adapted to the background filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$. Therefore, although the details of the derivation slightly change, one can show that the pricing formula given in Eq. (3.2.6) can still be applied in the same way once we use the G-intensities instead, since now we can write all the processes involved in the formula adapted to the background filtration.

### A.2 Origin of the Funding Spread $y^{(i,k)}$ in the Pricing Formula

Here, let us comment about the origin of the funding spread $y$ in our pricing formula in Eq. (3.4.2). Consider the following hypothetical but plausible situation to get a clear image:

(1): An interest rate swap market where the participants are discounting future cash flows by domestic OIS rate, regardless of the collateral currency, and assume there is no price dispute among them. (2): Party 1 enters two opposite trades with party 2 and 3, and they are agree to have CSA which forces party 2 and 3 to always post a domestic currency $U$ as collateral, but party 1 is allowed to use a foreign currency $E$ as well as $U$. (3): There is very liquid CCOIS market which allows firms to enter arbitrary length of swap. The spread $y$ is negative for CCOIS between $U$ and $E$, where $U$ is a base currency (such as USD in the above explanation).

In this example, the party 1 can definitely make money. Suppose, at a certain point, the party 1 receives $N$ unit amount of $U$ from the party 2 as collateral. Party 1 enters a CCOIS as spread payer, exchanging $N$ unit amount of $U$ and the corresponding amount of $E$, by which it can finance the foreign currency $E$ by the rate of ($E$’s OIS + $y^{(E,U)}$). Party 1 also receives $U$’s OIS rate from the CCOIS counterparty, which is going to be paid as the collateral margin to the party 2. Party 1 also posts $E$ to the party 3 since it has opposite position, it receives $E$’s OIS rate as the collateral margin from the party 3. As a result, the party 1 earns $-y^{(E,U)} (> 0)$ on the notional amount of collateral. It can rollover the CCOIS, or unwind it if $y$’s sign flips.

Of course, in the real world, CCS can only be traded with certain terms which makes the issue not so simple. However, considering significant size of CCS spread (a several tens of bps) it still seems possible to arrange appropriate CCS contracts to achieve cheaper funding. For a very short term, it may be easier to use FX forward contracts (or FX swaps) for the same purpose. In order to prohibit this type of arbitrage, party 1 should pay extra premium to make advantageous CSA contracts. This is exactly the reason why our pricing formula contains the funding spread $y$. 

77
A.3 Calibration to swap markets

For the details of calibration procedures, the numerical results and recent historical behavior of underlyings are available in Refs. [10, 12]. The procedures can be briefly summarized as follows: (1) Calibrate the forward collateral rate \( c(i)(0, t) \) for each currency using OIS market. (2) Calibrate the forward Libor curves by using the result of (1), IRS and tenor swap markets. (3) Calibrate the forward \( y(i,j)(0, t) \) spread for each relevant currency pair by using the results of (1), (2) and CCS markets.

Although we can directly obtain the set of \( y(i,j) \) from CCS, we cannot uniquely determine each \( y(i) \), which is necessary for the evaluation of Gateaux derivative when we deal with unilateral collateralization and CCA (collateral cost adjustment). For these cases, we need to make an assumption on the risk-free rate for one and only one currency. For example, if we assume that ON rate and the risk-free rate of currency \( (j) \) are the same, and hence \( y(j) = 0 \), then the forward curve of \( y(USD) \) is fixed by \( y(USD)(0, t) = -y(j,USD)(0, t) \).

Using Gateaux derivative, we can approximate the contract price as

\[ V_0 \simeq V_0 + CCA_0 \]

(A.4.6)

where

\[ CCA_0 = E^{Q(j)} \left[ \int_0^{T_N} e^{-\int_0^u (c(u) + y(u))du} \max(-\nabla_s, 0) \max(-y(j,i), 0) ds \right]. \]

(A.4.7)
Although $\nabla_t$ is simply a price under symmetric collateralization using currency $(i)$, we need to be careful about the advance reset conventions. One can show that

$$
\nabla_t = \sum_{n=\gamma(t)+1}^{N} E^{Q(j)} \left[ e^{-\int_t^{T_n} (c_u^{(j)} + y_u^{(j,i)}) du} \left( -e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - \delta_n B + \frac{f_x^{(j,i)}}{f_x^{(j,i)}} e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} \right) \right] F_t 
+ E^{Q(j)} \left[ e^{-\int_t^{T_{\gamma(t)-1}} (c_u^{(j)} + y_u^{(j,i)}) du} \left( -e^{\int_{T_{\gamma(t)-1}}^{T_{\gamma(t)}} c_{\gamma(t)}^{(j)}} du - \delta_{\gamma(t)} B + \frac{f_x^{(j,i)}}{f_x^{(j,i)}} e^{\int_{T_{\gamma(t)-1}}^{T_{\gamma(t)}} c_{\gamma(t)}^{(j)} du} f_x^{(j,i)}(T_{\gamma(t)}) \right) \right] F_t,
$$

(A.4.8)

where $\gamma(t) = \min\{n; T_n > t, n = 1 \cdots N\}$. Note that $T_{\gamma(t)-1} < t$ since we are considering spot-start swap (or $T_0 = 0$). Assuming the independence of $y^{(j,i)}$ and other variables, we can simplify $V_t(0)$ and obtains

$$
\nabla_t = -\sum_{n=\gamma(t)}^{N} D^{(j)}(t, T_n) Y^{(j,i)}(t, T_n) \delta_n B + \sum_{n=\gamma(t)+1}^{N} D^{(j)}(t, T_{n-1}) \left( Y^{(j,i)}(t, T_{n-1}) - Y^{(j,i)}(t, T_n) \right) 
- Y^{(j,i)}(t, T_{\gamma(t)}) e^{\int_{T_{\gamma(t)}}^{T_{\gamma(t)-1}} c_{\gamma(t)}^{(j)} ds} + \frac{\int_x^{(j,i)}(T_{\gamma(t)-1})}{f_x^{(j,i)}(T_{\gamma(t)-1})} e^{\int_{T_{\gamma(t)-1}}^{T_{\gamma(t)}} c_{\gamma(t)}^{(j)} ds},
$$

(A.4.9)

where we have defined $Y^{(j,i)}(t, T) = E^{Q(j)} \left[ e^{-\int_t^T y_s^{(j,i)} ds} \bigg| F_t \right]$. We need to evaluate the above $\nabla$ at each time step of Monte Carlo simulation to calculate CCA of Eq. (A.4.7).

### A.4.2 Asymmetrically collateralized OIS

For spot-start, $T_N$-maturing OIS, we have

$$
V_0 = E^{Q(j)} \left[ \int_{[0,T_N]} e^{-\int_0^s R(t,V_u) du} dD_s \right], \quad (A.4.10)
$$

where

$$
dD_s = \sum_{n=1}^{N} \delta_{T_n}(s) \left[ \delta_n S - \left( e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - 1 \right) \right], \quad (A.4.11)
$$

and

$$
R(t, V_t) = c_t^{(j)} + \max(y_t^{(j,i)}, 0) \mathbf{1}_{\{V_t < 0\}}. \quad (A.4.12)
$$

Using Gateaux derivative, the above swap value can be approximated as

$$
V_0 \approx \nabla_0 + \text{CCA}_0, \quad (A.4.13)
$$

where

$$
\text{CCA}_0 = E^{Q(j)} \left[ \int_0^T e^{-\int_0^s c_u^{(j)} du} \max(-\nabla_s, 0) \max(y_s^{(j,i)}, 0) ds \right], \quad (A.4.14)
$$

79
V_t = E^Q \left[ \sum_{n=\gamma(t)}^{N} e^{-\int_{r_n}^{r_{n-1}} c_u^{(j)} du} \left( \delta_n S - \left( e^{\int_{r_n}^{r_{n-1}} c_u^{(j)} du} - 1 \right) \right) \bigg| \mathcal{F}_t \right] \\
= \sum_{n=\gamma(t)}^{N} D^{(j)}(t, T_n) \delta_n S - e^{\int_{r_{\gamma(t)-1}}^{r_{\gamma(t)}} c_u^{(j)} du} + D^{(j)}(t, T_N). \quad \text{(A.4.15)}

Here, S is the fixed OIS rate.

### A.5 Proof of Proposition 2

Consider the case of \( y_1 \geq y_2 \). From Eq. (3.2.6), one can show that the pre-default value \( V \) can also be written in the following recursive form:

\[
V_t = E^Q \left[ -\int_{[t,T]} \left( r_s - \mu(s, V_s) \right) V_s ds + \int_{[t,T]} dD_s \bigg| \mathcal{F}_t \right]. \quad \text{(A.5.1)}
\]

Let us define the following variables:

\[
\tilde{V}_t = e^{-\int_0^t (r_s - y_1) ds} V_t \quad \text{(A.5.2)}
\]

\[
\tilde{D}_t = \int_{[0,t]} e^{-\int_s^t (r_u - y_1) du} dD_u. \quad \text{(A.5.3)}
\]

Note that

\[
 r_t - \mu(t, V_t) = (r_t - y_1) + (y_1 - y_2) \mathbf{1}_{\{V_t \geq 0\}} = (r_t - y_1) + \eta^{1,2}_t \mathbf{1}_{\{V_t \geq 0\}}, \quad \text{(A.5.4)}
\]

where we have defined \( \eta^{i,j} = y^i - y^j \). Using new variables, Eq. (A.5.1) can be rewritten as

\[
\tilde{V}_t = E^Q \left[ -\int_{[t,T]} \eta^{1,2}_s \mathbf{1}_{\{\tilde{V}_s \geq 0\}} \tilde{V}_s ds + \int_{[t,T]} d\tilde{D}_s \bigg| \mathcal{F}_t \right]. \quad \text{(A.5.5)}
\]

And hence we have,

\[
\tilde{V}_t^{ab} - \tilde{V}_t^{a} - \tilde{V}_t^{b} = E^Q \left[ -\int_{[t,T]} \eta^{1,2}_s \left( \max(V^{ab}_s, 0) - \max(V^{a}_s, 0) - \max(V^{b}_s, 0) \right) ds \bigg| \mathcal{F}_t \right]. \quad \text{(A.5.6)}
\]

Let us denote the upper bound of \( \eta^{1,2} \) as \( \alpha \), and also define \( Y = \tilde{V}^{ab} - \tilde{V}^{a} - \tilde{V}^{b} \) and \( G_s = -\eta^{1,2}_s \left( \max(V^{ab}_s, 0) - \max(V^{a}_s, 0) - \max(V^{b}_s, 0) \right) \). Then, we have \( Y_T = 0 \) and

\[
Y = E^Q \left[ \int_{[t,T]} G_s ds \bigg| \mathcal{F}_t \right]. \quad \text{(A.5.7)}
\]
\[ G_s = -\eta_s^{1,2} \left( \max(\tilde{V}_s^{ab}, 0) - \max(\tilde{V}_s^a, 0) - \max(\tilde{V}_s^b, 0) \right) \]
\[ \geq -\eta_s^{1,2} \left( \max(\tilde{V}_s^{ab}, 0) - \max(\tilde{V}_s^a + \tilde{V}_s^b, 0) \right) \]
\[ \geq -\eta_s^{1,2} \max(\tilde{V}_s^{ab} - \tilde{V}_s^a - \tilde{V}_s^b, 0) \]
\[ \geq -\alpha |Y_s| . \]  

(A.5.8)

Applying the consequence of the Stochastic Gronwall-Bellman Inequality in Lemma B2 of Ref. [6] to \( Y \) and \( G \), we can conclude \( Y_t \geq 0 \) for all \( t \in [0, T] \), and hence \( V^{ab} \geq V^a + V^b \).

A.6 Proof of Proposition 3

Consider the case of \( y^1 \geq y^2 \). Let us define
\[ \tilde{V}_t^F = e^{-\int_t^0 (r_s - y_s^1) ds} V_t^F \]
\[ \tilde{V}_t^G = e^{-\int_t^0 (r_s - y_s^1) ds} V_t^G , \]
as well as
\[ \tilde{D}_t = \int_{[0,t]} e^{-\int_0^s (r_u - y_u^1) du} dD_s \]
as in the previous section. Then, we have
\[ \tilde{V}_t^G = \mathbb{E}^Q \left[ -\int_t^T \eta_s^{1,2} \max(\tilde{V}_s^G, 0) ds + \int_t^T d\tilde{D}_s \mid G_t \right] \]  

(A.6.4)
\[ \tilde{V}_t^F = \mathbb{E}^Q \left[ -\int_t^T \eta_s^{1,2} \max(\tilde{V}_s^F, 0) ds + \int_t^T d\tilde{D}_s \mid F_t \right] . \]  

(A.6.5)

Now, let us define
\[ U_t = \mathbb{E}^Q \left[ \tilde{V}_t^G \mid F_t \right] . \]  

(A.6.6)

Then, using Jensen’s inequality, we have
\[ U_t \leq \mathbb{E}^Q \left[ -\int_t^T \eta_s^{1,2} \max(U_s, 0) ds + \int_t^T d\tilde{D}_s \mid F_t \right] . \]  

(A.6.7)

Therefore, we obtain
\[ \tilde{V}_t^F - U_t \geq \mathbb{E}^Q \left[ -\int_t^T \eta_s^{1,2} \left( \max(\tilde{V}_s^F, 0) - \max(U_s, 0) \right) ds \mid F_t \right] \]  

(A.6.8)
\[ \geq \mathbb{E}^Q \left[ -\int_t^T \eta_s^{1,2} |\tilde{V}_s^F - U_s| ds \mid F_t \right] . \]  

(A.6.9)

Using the stochastic Gronwall-Bellman Inequality as before, one can conclude that \( \tilde{V}_t^F \geq U_t \) for all \( t \in [0, T] \), and in particular, \( V_0^F \geq V_0^G \).
A.7 Comparison of Gateaux Derivative with PDE

In order to get clear image for the reliability of Gateaux derivative, we compare it with the numerical result directly obtained from PDE. We consider a simplified setup where MtMCCOIS exchanges the coupons continuously, and the only stochastic variable is a spread \( y \). Consider continuous payment \((i, j)\)-MtMCCOIS where the leg of currency \((i)\) needs notional refreshments. We assume following situation as the asymmetric collateralization:

1. Party 1 is the basis spread payer and can use either the currency \((i)\) or \((j)\) as collateral.
2. Party 2 is the basis spread receiver and can only use the currency \((i)\) as collateral.

In this case, one can see that the value of \( t \)-start \( T \)-maturing contract from the viewpoint of party 1 is given by (See, Eq. (3.5.17).)

\[
V_t = \mathbb{E}^{Q(j)} \left[ \int_t^T \exp \left( - \int_t^s R(u, V_u)du \right) \left( y^{(j,i)}_s - B \right) ds \mid \mathcal{F}_t \right], \tag{A.7.1}
\]

where

\[
R(t, V_t) = c^{(j)}(t) + y^{(j,i)}_t + \max \left( -y^{(j,i)}_t, 0 \right) \mathbf{1}_{\{V_t < 0\}} \tag{A.7.2}
\]

and \( B \) is a fixed spread for the contract. \( y^{(j,i)} \) is the only stochastic variable and its dynamics is assumed to be given by the following Hull-White model:

\[
dy^{(j,i)}_t = \left( \theta^{(j,i)}(t) - \kappa^{(j,i)} y^{(j,i)}_t \right) dt + \sigma^{(j,i)}_y dW^{Q(j)}_t. \tag{A.7.3}
\]

Here, \( \theta^{(j,i)}(t) \) is a deterministic function specified by the initial term structure of \( y^{(j,i)} \), \( \kappa^{(j,i)} \) and \( \sigma^{(j,i)}_y \) are constants. \( W^{Q(j)} \) is a Brownian motion under the spot martingale measure of currency \((j)\).

The PDE for \( V_t \) is given by

\[
\frac{\partial}{\partial t} V(t, y) + \left( \gamma(t, y) \frac{\partial V(t, y)}{\partial y} + \frac{\sigma^{(j,i)}_y}{2} \frac{\partial^2 V(t, y)}{\partial y^2} \right) - R(t, V(t, y)) V(t, y) + y - B = 0 , \tag{A.7.4}
\]

where

\[
\gamma(t, y) = \theta^{(j,i)}(t) - \kappa^{(j,i)} y. \tag{A.7.5}
\]

If party 1 is a spread receiver, we need to change \( y - B \) to \( B - y \), of course.

Terminal boundary condition is trivially given by \( V(T, \cdot) = 0 \). On the lower boundary of \( y \) or when \( y = -M \) \((= y_{\text{min}}) \ll 0\), we have \( V_t < 0 \) for all \( t \). Thus, we have \( R(s, V(s, y)) = c^{(j)}(s) \) for all \( s \geq t \), if \( y = -M \) at time \( t \). Therefore, on the lower boundary, the value of MtMCCOIS is given by

\[
V(t, -M) = \mathbb{E}^{Q(j)} \left[ \int_t^T e^{-\int_t^s c^{(j)}(u)du} (y^{(j,i)}_s - B) ds \mid y^{(j,i)}_t = -M \right] = \int_t^T D^{(j)}(t, s) \left( -B - \frac{\partial}{\partial s} \ln Y^{(j,i)}(t, s) \right) ds. \tag{A.7.6}
\]

82
Since \( c^{(j)}(t) \) is a deterministic function, \( D^{(j)}(t, s) = D^{(j)}(0, s)/D^{(j)}(0, t) \) is simply given by the forward.

On the other hand, when \( y = M (= y_{\text{max}}) \gg 0 \), we have \( V_t > 0 \) for all \( t \). Thus we have \( R(s, V(s, y)) = c^{(j)}(s) + y^{(j,i)}(s) \) for all \( s \geq t \), if \( y = M \) at time \( t \). Thus, on the upper boundary, the value of the contract becomes

\[
V(t, M) = E^{Q(j)} \left[ \int_t^T e^{-\int_t^s (c_u^{(j)} + y_u^{(j,i)})du} \left( y_s^{(j,i)} - B \right) \left| Y(t, s) = M \right. \right] \\
= \int_t^T \left\{ -BD^{(j)}(t, s)Y^{(j,i)}(t, s) - D^{(j)}(t, s) \frac{\partial}{\partial s} Y^{(j,i)}(t, s) \right\} ds .
\] (A.7.7)

Now let us compare the numerical result between Gateaux derivative and PDE. In the case of Gateaux derivative, the contract value is approximated as

\[
V_t \simeq \nabla V_t + \nabla V_t,
\] (A.7.8)

where

\[
\nabla V_t = E^{Q(j)} \left[ \int_t^T e^{-\int_t^s (c_u^{(j)} + y_u^{(j,i)})du} \left( y_s^{(j,i)} - B \right) \left| \mathcal{F}_t \right. \right] ,
\] (A.7.9)

and

\[
\nabla V_t = E^{Q(j)} \left[ \int_t^T e^{-\int_t^s (c_u^{(j)} + y_u^{(j,i)})du} \left[ -\nabla V + \max(-y_s^{(j,i)}, 0) \right] ds \left| \mathcal{F}_t \right. \right] .
\] (A.7.10)

\( V_t \) is the value of the contract under symmetric collateralization where both parties post currency \((i)\) as collateral, and \( \nabla V_t \) is a deviation from it.

In Fig. A.1, we plot the price difference of continuous 10y-MtMCCOIS from its symmetric limit obtained by PDE and Gateaux derivative with various volatility of \( y^{(j,i)} \). Term structures of \( y^{(j,i)} \) and other curves are given in Appendix A.8. Here, the spread \( B \) is chosen in such a way that the swap price is zero in the case where both parties can only use currency \((i)\) as collateral, or \( B \) is a market par spread. The price difference is \( V_t - \nabla V_t \) and expressed as basis points of notional. From our analysis using the recent historical data in Ref. [12], we know that the annualized volatility of \( y \) is around 50 bps for a calm market but it can be more than \((100 \sim 200)\) bps when CCS market is volatile (We have used EUR/USD and USD/JPY pairs.). One observes that Gateaux derivative provides reasonable approximation for wide range of volatility. If the party 1 is a spread receiver, both of the methods give very small price differences, less than 1bp of notional.

### A.8 Data used in Numerical Studies

The parameter we have used in simulation are

\[
\kappa^{(j)} = \kappa^{(i)} = 1.5\% \tag{A.8.1}
\]

\[
\sigma_c^{(j)} = \sigma_c^{(i)} = 1\% \tag{A.8.2}
\]

\[
\sigma_x^{(j,i)} = 12\% . \tag{A.8.3}
\]

All of them are defined in annualized term. The volatility of \( y^{(j,i)} \) is specified in the main text in each numerical analysis.
Figure A.1: Price difference from symmetric limit for 10y continuous MtMCCOIS

Term structures and correlation used in simulation are given in Fig. A.2. There we have defined

\[
R^{(k)}_{\text{OIS}}(T) = -\frac{1}{T} \ln E^{Q^{(k)}} \left[ e^{-\int_0^T c^{(k)}_s ds} \right]
\]

\[
R^{(j,i)}_{\text{y}}(T) = -\frac{1}{T} \ln E^{Q^{(j)}} \left[ e^{-\int_0^T y^{(j,i)}_s ds} \right].
\]

The curve data is based on the calibration result of typical JPY and USD market data of early 2010. In Monte Carlo simulation, in order to reduce simulation error, we have adjusted drift terms \( \theta(t) \) to achieve exact match to the relevant forwards in each time step.
Figure A.2: Term structures and correlation used for simulation

<table>
<thead>
<tr>
<th>Instantaneous correlation $\rho$</th>
<th>$c(j)$</th>
<th>$c(i)$</th>
<th>$r(x(i))$</th>
<th>$y(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(j)$</td>
<td>100%</td>
<td>24%</td>
<td>-5%</td>
<td>0%</td>
</tr>
<tr>
<td>$c(i)$</td>
<td>24%</td>
<td>100%</td>
<td>15%</td>
<td>0%</td>
</tr>
<tr>
<td>$r(x(i))$</td>
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<td>15%</td>
<td>100%</td>
<td>0%</td>
</tr>
<tr>
<td>$y(j)$</td>
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<td>0%</td>
<td>0%</td>
<td>100%</td>
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<table>
<thead>
<tr>
<th>OIS of currency ($)</th>
<th>OIS of currency ($)</th>
<th>$y(j)$ spread</th>
</tr>
</thead>
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<tr>
<td>term R OIS</td>
<td>term R OIS</td>
<td>term R O(i)</td>
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<tr>
<td>1d 0.8650%</td>
<td>0d 0.2527%</td>
<td>0d -0.1742%</td>
</tr>
<tr>
<td>1m 0.0643%</td>
<td>3m 0.2547%</td>
<td>1y -0.1905%</td>
</tr>
<tr>
<td>3m 0.0643%</td>
<td>1y 0.3086%</td>
<td>2y -0.2322%</td>
</tr>
<tr>
<td>6m 0.0865%</td>
<td>2y 0.8272%</td>
<td>3y -0.2628%</td>
</tr>
<tr>
<td>1y 0.0870%</td>
<td>3y 1.3838%</td>
<td>4y -0.2877%</td>
</tr>
<tr>
<td>18m 0.1060%</td>
<td>4y 1.8071%</td>
<td>5y -0.3014%</td>
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<tr>
<td>2y 0.1313%</td>
<td>5y 2.1577%</td>
<td>6y -0.3095%</td>
</tr>
<tr>
<td>3y 0.1865%</td>
<td>7y 2.6196%</td>
<td>7y -0.3123%</td>
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<tr>
<td>4y 0.2605%</td>
<td>10y 3.0411%</td>
<td>8y -0.3600%</td>
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<tr>
<td>5y 0.3883%</td>
<td>12y 3.1560%</td>
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<tr>
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<td>7y 0.6584%</td>
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<tr>
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<tr>
<td>30y 1.9627%</td>
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Bibliography


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